Graz University of Technology

Department of Mathematical Structure Theory

Diskrete Stochastik und Informationstheorie
Exercise sheet 3 – 18/4/2013

Entropy of a random variable

Exercise 11 *A fair coin is flipped until the first head occurs. Let X denote the number of flips required.*

1) *Calculate the entropy $H(X)$ in bits. (Hint: Think about $\sum_{n=0}^{\infty} x^n = 1/(1-x)$ and deriving.)*

2) *A random variable X is drawn according to this distribution. Find an "efficient" sequences of yes-no questions of the form "Is X contained in the set S?" Compare $H(X)$ to the expected number of questions required to determine the value of X.*

Exercise 12 *Let X be a random variable taking only a finite number of values in $\mathbb{R}$. Which of the (in)-equalities $H(X) \leq H(Y)$, $H(X) = H(Y)$, $H(X) \geq H(Y)$ does hold in general if*

1) $Y = e^X$?

2) $Y = |X|$?

Exercise 13 *Let $\mathcal{P} = \{(p_1, \ldots, p_n) \,|\, p_1, \ldots, p_n \geq 0, \sum_{i=1}^{n} p_i = 1\}$ be the space of all n dimensional probability vectors.*

1) *What is the minimum entropy $\min\{H(\mathbf{p}) : \mathbf{p} \in \mathcal{P}\}$.*

2) *Find all $\mathbf{p} \in \mathcal{P}$ that achieve this minimum.*

Exercise 14 *Show that if $H(Y|X) = 0$, then Y is a function of X, i.e., for all x with $p(x) > 0$, there is only one possible value of y such that $p(x, y) > 0$.*

Exercise 15 *Suppose that one has n coins, among which there may or may not be one counterfeit coin. If there is a counterfeit coin, it may be either heavier or lighter than the other coins. The coins are to be weighed by a (two-pan) balance.*

1) *Find an upper bound on the number of coins n such that k weighings will find the counterfeit count (if any) and correctly declare it to be heavier or lighter.*

2) *What is a coin weighing strategy for $k = 3$ weighings and 12 coins?*