Graz University of Technology

Department of Mathematical Structure Theory

Diskrete Stochastik und Informationstheorie
Exercise sheet 5 – 9/5/2013

Entropy rate of stochastic processes

Exercise 21 *Consider the Markov chain $(X_n)_{n\geq 0}$ on $\mathcal{X} = \{1, \ldots, 7\}$ with transition matrix*

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

1) Is X_n irreducible?

2) Give a stationary distribution for X_n ? Is the latter unique?

3) Calculate $\mathbb{P}(X_n = i | X_0 = 6)$ for $i \in E$ and $n \in \{1, 2, 3\}$

4) Calculate $\lim_{n\to\infty} \mathbb{P}(X_n = i | X_0 = 6)$ for all $i \in E$.

Exercise 22 *Consider all simple random walks (nearest-neighbour, equal probability to go to each neighbour) on (undirected) graphs of 4 vertices.*

1) Which random walk (graph) has the highest, respectively lowest entropy?

2) What happens, if we restrict ourselves to connected graphs?

3) What happens if we allow loops (that is staying at the vertex is an option, too)?

Exercise 23 *Let $(X_n)_{n\in\mathbb{N}}$ be an irreducible Markov chain on the countable state space $\mathcal{X}$ with transition matrix P and stationary distribution ν. Form the associated transition process $(Y_n)_{n\in\mathbb{N}}$, which encodes the transitions effectuated by X, i.e. $Y_n = (X_n, X_{n+1})$:*

$$X(\omega) = (1, 3, 4, 5, 2, 4, \ldots) \quad \Rightarrow \quad Y(\omega) = ((1, 3), (3, 4), (4, 5), (5, 2), (2, 4), \ldots).$$

1) Show that Y is an irreducible Markov chain on $\mathcal{X}^2$ and state its transition matrix.

2) Calculate its invariant distribution.

3) Calculate its entropy.

Exercise 24 *Show, for a Markov chain, that*

$$H(X_0|X_n) \geq H(X_0|X_{n-1}).$$