Diskrete Stochastik und Informationstheorie
Exercise sheet 6 – 16/5/2013

Entropy rate and AEP

Exercise 25 *A particle walks on the integers in the direction he is facing (left or right), reversing direction after each step taken with probability $p = 0.1$. The walker starts at 0 facing to the right.*

1) *Show that X is a Markov chain and describe its state space and transition probabilities.*
2) *What is the expected number of steps taken by the walker before reversing direction?*
3) *Find $H(X_1, \dots, X_n)$.*
4) *Find the entropy of this walker.*

Exercise 26 *Let X_i be i.i.d. random variables with values in $\mathcal{X} = \{1, 2, \dots, m\}$ and distribution $p(x)$. Denote $\mu = \mathbb{E}X$ and $H = -\sum p(x) \log p(x)$. Let $\epsilon > 0$ and*

$$A^n = \{x^n \in \mathcal{X}^n : |-\frac{1}{n} \log p(x^n) - H| \leq \epsilon\}$$

and define

$$B^n = \{x^n \in \mathcal{X}^n : |\frac{1}{n} \sum_{i=1}^{n} X_i - \mu| \leq \epsilon\}.$$

1) *Does $\mathbb{P}((X_1, \dots, X_n) \in A^n) \to 1$?*
2) *Does $\mathbb{P}((X_1, \dots, X_n) \in A^n \cap B^n) \to 1$?*
3) *Show that $|A^n \cap B^n| \leq 2^{n(H+\epsilon)}$ for all n.*
4) *Show that $|A^n \cap B^n| \geq (\frac{1}{2}) 2^{n(H-\epsilon)}$ for all n sufficiently large.*

Exercise 27 *A discrete memoryless source emits a sequence of statistically independent binary bits with probabilities $p(1) = 0.005$ and $p(0) = 0.995$. The digits are taken 100 at a time and a binary codeword is provided for every sequence of 100 digits containing three or fewer 1's.*

1) *Assuming that all codewords are the same length, find the minimum length required to provide codewords for all sequences with three or fewer 1's.*
2) *Calculate the probability of observing a source sequence for which no codeword has been assigned.*
3) *Use Chebyshev's inequality to bound the probability of observing a source sequence for which no codeword has been assigned. Compare this bound with the probability obtained in 2).*

Exercise 28 *Let*

$$X = \begin{cases} 1, & \textit{with probability } \frac{1}{2} \\ 2, & \textit{with probability } \frac{1}{3} \\ 3, & \textit{with probability } \frac{1}{6} \end{cases}$$

Let $X_1, X_2, \dots$ be drawn i.i.d. according to the distribution of X. Find the limiting behavior of $(X_1 X_2 \cdots X_n)^{1/n}$ as $n \to \infty$.