Kazhdan Constants for Conjugacy Classes of Compact Groups

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Abstract

Kazhdan constants relative to conjugacy classes of compact groups are computed. They depend on the nontrivial irreducible characters of the respective group. The result is applied in particular to finite groups of Lie type, symmetric groups, and the group SU(n).

1 Introduction and statement of the results

Property (T) introduced by D. A. Kazhdan in 1967 in [16] is a geometric property of the representations of a group. It plays an important role for groups like lattices in semi-simple Lie groups. A survey on Property (T) can be found in [12].

Let $G$ be a locally compact group and let $\pi$ be a strongly continuous unitary representation of $G$ on a Hilbert space $H_{\pi}$. The Kazhdan constant associated to $\pi$ and a compact subset $Q$ of $G$ is defined by

$$K(\pi, G, Q) = \inf_{\xi \in H_{\pi}^1} \sup_{g \in Q} \|\pi(g)\xi - \xi\|,$$

where $H_{\pi}^1$ is the unit sphere in $H_{\pi}$. The absolute Kazhdan constant relative to a compact subset $Q$ is

$$K(G, Q) = \inf_{\pi \in \mathcal{H}} K(\pi, G, Q),$$
where $R$ is the set of equivalence classes of representations of $G$ on separable Hilbert spaces not containing the trivial representation. The group $G$ has Property (T), if there exists some compact $Q$ of $G$ such that $K(G, Q) > 0$.

The problem of computing explicit Kazhdan constants occurs as a natural question in [12, page 133]. These constants provide a quantitative version of Property (T) and can be seen as measuring a kind of distance of the trivial representation to those not containing it. They play an important role in different applications of groups with Property (T). Among those are for example the explicit construction of expanding graphs, see [19], or more recent applications to the product replacement algorithm, see [20], or to random walks, see [22]. In the last-mentioned application where finitely generated groups are considered it is of greater importance to consider the expression

$$
\bar{K}(G, Q) = \inf_{\pi \in R} \inf_{\xi \in H} \frac{1}{|Q|} \sum_{g \in Q} \|\pi(g) \xi - \xi\|^2
$$

which focuses on the average over the generating set and not the maximum. From [22, page 1893] (or [6, page 36] in the special case of groups with symmetric presentation) it can be deduced that for conjugacy classes of finite groups these two are in fact equal where in general only $K(G, Q) \geq \bar{K}(G, Q)$ holds.

It is trivial to show that compact groups have Property (T). On the other hand the computation of Kazhdan constants is nontrivial even for this class of groups. For this see for example [2], [5], and [7]. A more recent reference is [3] which considers another variant of the Kazhdan constant where $R$ is the smaller set consisting only of nontrivial irreducible representations. These two constants can in fact be different as was observed in [2, page 496]. On the other hand for $\bar{K}$ it would be enough to consider only nontrivial irreducible representations as was observed in [6, page 21]. For computations of exact Kazhdan constants in particular for non-compact groups see for example [4], [6], [12], and [27]. Especially [27] contains further references for computations of estimates for Kazhdan constants.

The purpose of this paper is to determine constants associated to a conjugacy class $Q$ for compact groups. In general there is no simple formula for the Kazhdan constant.

Our method is based on the following result, which allows us to restrict ourselves to the consideration of irreducible characters.

**Theorem 1.1** Let $G$ be a compact group and $Q$ a conjugacy class of $G$. Then

$$
K(G, Q) = \inf_{\chi \in Z} \sqrt{2 - \frac{2}{m_\chi} \Re \chi(Q)},
$$

where $Z$ is the set of irreducible characters of $G$. Theorem 1.1
where $Z$ is the set of all nontrivial irreducible characters of $G$ and $m_\chi = \chi(1)$ is the degree of $\chi$.

As a first application, we determine lower bounds for Kazhdan constants of finite groups of Lie type.

In Section 3, it will be shown that if $Q$ is a compact subset not generating a dense subgroup, then $K(G,Q) = 0$. This shows for simple groups that nontrivial conjugacy classes always yield positive constants as they generate a normal subgroup.

We shall also deal with the symmetric group $S_n$ for which we prove the following result.

**Theorem 1.2** Let $n \geq 2$ and $Q$ be a conjugacy class in $S_n$ with $\text{sgn} Q = -1$, we have

$$K(S_n, Q) \geq \frac{2}{\sqrt{n - 1}},$$

with equality if $Q$ is the conjugacy class of all 2-cycles.

It was shown in [2] that the exact Kazhdan constant for $S_n$ relative to the generating set $\{(1,2), (2,3), \ldots, (n-1,n)\}$ is $\sqrt{24/(n^3 - n)}$. As the conjugacy class of all 2-cycles contains this set, it follows from this result that

$$K(S_n, Q) \geq \sqrt{\frac{24}{(n^2 + n)(n - 1)}}.$$

Finally, in the case of the special unitary group $G = SU(n)$, we give the following lower bound for the Kazhdan constant.

**Theorem 1.3** For $G = SU(n)$ and the conjugacy class $Q_t$ determined by the eigenvalues $e^{it_1}, \ldots, e^{it_n}$

$$K(G, Q_t) \geq \sqrt{\frac{2}{n} \max_{1 \leq r,s \leq n} \left| \sin \left( \frac{t_r - t_s}{2} \right) \right|}.$$

Moreover the exact Kazhdan constant for $SU(2)$ is also determined.
2 Kazhdan constants relative to a conjugacy class

In this section, we determine lower and upper bounds for the Kazhdan constant with respect to a representation relative to a conjugacy class which will prove Theorem 1.1. The bound depends on the nontrivial irreducible characters of the compact group.

The following is an important step for the proof of Theorem 1.1 and relies on the Schur orthogonality relations for compact groups, see for example [10, pages 129–130]. In the following integration will always be with respect to normalized Haar measure.

**Lemma 2.1** Let $G$ be a compact group and $q \in G$. Let $\pi$ be an irreducible representation of $G$ on $H_\pi$, $\xi \in H_\pi$, $m = \dim H_\pi$, and $\chi = \text{tr} \pi$. Then

$$\int_G \| \pi(h^{-1}qh) \xi - \xi \|^2 \, dh = \left( 2 - \frac{2}{m} \text{Re} \chi(q) \right) \| \xi \|^2.$$

**PROOF.** As $\pi(q)$ is unitary there exists an orthonormal basis $e_1, \ldots, e_m \in H_\pi$ with respect to which $\pi(q)$ is diagonal. Then the scalar product can be expanded in this basis as

$$\langle \pi(qh) \xi, \pi(h) \xi \rangle = \sum_{s=1}^m \langle \pi(h) \xi, \pi(q^{-1}) e_s \rangle \langle e_s, \pi(h) \xi \rangle = \sum_{s=1}^m |\langle \pi(h) \xi, e_s \rangle|^2 \langle \pi(q) e_s, e_s \rangle$$

using the diagonality of $\pi(q)$ in this basis. Now the integral can be computed using the Schur orthogonality relations and the above

$$\int_G \| \pi(h^{-1}qh) \xi - \xi \|^2 \, dh = 2 \| \xi \|^2 - 2 \text{Re} \int_G \sum_{s=1}^m |\langle \pi(h) \xi, e_s \rangle|^2 \langle \pi(q) e_s, e_s \rangle \, dh = \left( 2 - \frac{2}{m} \text{Re} \chi(q) \right) \| \xi \|^2. \quad \Box$$

The following will prove Theorem 1.1.

**Proposition 2.2** Let $G$ be a compact group, $Q$ a conjugacy class of $G$, and $\pi$ a representation of $G$. Let $Z$ be the set of all irreducible characters $\chi$ of $G$
associated with the irreducible components of the representation \( \pi \) with degrees \( m_\chi = \chi(1) \) and \( Z' \subset Z \) the set of all such \( \chi \) with \( m_\chi \chi \) contained in \( \pi \). Then

\[
\inf_{\chi \in Z} \sqrt{2 - \frac{2}{m_\chi} \Re \chi (Q)} \leq K(\pi, G, Q) \leq \inf_{\chi \in Z'} \sqrt{2 - \frac{2}{m_\chi} \Re \chi (Q)}.
\]

**PROOF.** At first the lower bound is shown. As \( G \) is a compact group, there are irreducible representations \( \pi_r \) of \( G \) on \( H_r \) such that \( H_\pi = \bigoplus_r H_r \), \( \pi = \bigoplus_r \pi_r \), and \( \chi_r = \text{tr} \pi_r \in Z \). Let \( g \in Q \), then

\[
\| \pi(g) \xi - \xi \|^2 = \sum_r \| \pi_r(g) \xi_r - \xi_r \|^2,
\]

where \( \xi = \sum \xi_r \) with \( \xi_r \in H_r \) for all \( r \) and \( \xi \in H_\pi^1 \). Let \( q \) be a representative of the conjugacy class \( Q \) and \( m_r = \dim H_r \). Then bounding the supremum from below by integrating, using the previous lemma, and taking the infimal value

\[
\sup_{g \in Q} \| \pi(g) \xi - \xi \|^2 \geq \int_G \sum_r \| \pi_r(h^{-1}qh) \xi_r - \xi_r \|^2 \, dh \geq \inf_r \left( 2 - \frac{2}{m_r} \Re \chi_r(q) \right).
\]

Hence putting the preceding results into the formula for the Kazhdan constant

\[
K(\pi, G, Q) \geq \inf_{\chi \in Z} \sqrt{2 - \frac{2}{m_\chi} \Re \chi (Q)}.
\]

Now the upper bound is considered. Let \( \chi \in Z' \) and \( \sigma \) an irreducible representation of \( G \) on \( H_\sigma \) with \( \chi = \text{tr} \sigma \). Then there exists a subrepresentation of \( \pi \) which is isomorphic to \( \sigma \otimes \text{id}_{H_\sigma} \) with character \( m_\chi \chi \). Let \( e_1, \ldots, e_{m_\chi} \) be an orthonormal basis of \( H_\sigma \) and \( \xi = m_\chi^{-1/2} \sum_{s=1}^{m_\chi} e_s \otimes e_s \in H_\sigma \otimes H_\sigma \). The matrix coefficient gives now the desired upper bound

\[
\| (\sigma \otimes \text{id}_{H_\sigma})(g) \xi - \xi \|^2 = 2 - \frac{2}{m_\chi} \Re \sum_{r,s=1}^{m_\chi} \langle (\sigma \otimes \text{id}_{H_\sigma})(g)(e_r \otimes e_r), e_s \otimes e_s \rangle
\]

\[
= 2 - \frac{2}{m_\chi} \Re \sum_{r,s=1}^{m_\chi} \langle \sigma(g)e_r, e_s \rangle \langle e_r, e_s \rangle = 2 - \frac{2}{m_\chi} \Re \chi(g).
\]

The proposition also shows \( K(G, Q) = K(\lambda, G, Q) \) for a conjugacy class \( Q \).
where $\lambda$ is the regular representation restricted to the orthogonal complement of the constant functions in $L^2(G)$.

**Example 2.3** Let $C_n < T$ be the cyclic group of order $n$, where $T$ denotes the circle group $T = \{c \in \mathbb{C} : |c| = 1\}$. The set of all irreducible representations consists of $\chi_k$ for $0 \leq k \leq n - 1$, where $\chi_k(c) = c^k$ for $c \in C_n$. Hence

$$K(C_n, \{c\}) = \inf_{1 \leq k \leq n-1} \sqrt{2 - 2 \cos (2\pi k/n)} = 2 \sin (\pi/n)$$

for the generating element $c = e^{2\pi i/n}$; see also [12, page 16] or [6, page 19].

There seem to be no computations of the quantity $\sup_{\chi \in \mathbb{Z}} m_{\chi}^{-1} \operatorname{Re} \chi(Q)$ in the literature, but there are several sources giving upper bounds for the (obviously larger) $m_{\chi}^{-1} |\chi(Q)|$. Among those are for example [9], [21], and [25] for symmetric groups, [11] and [13] for finite groups of Lie type, [18] for arbitrary finite non-abelian simple groups, and [23] and [24] for compact Lie groups. In particular from [11] the following estimates for Kazhdan constants of finite groups of Lie type can be deduced. Upper bounds for normalized characters of these groups appear also in [13] but as they are stated there they are not as easily applicable to our result as those of [11].

**Theorem 2.4** Let $G$ be a quasi-simple and simply connected group of Lie type over a finite field with $k$ elements and $Q$ a conjugacy class of a unipotent element $\neq 1$, then

$$K(G, Q) \geq \sqrt{2 - \frac{2}{\sqrt{k} - 1}}$$

for $k \geq 5$ and

$$K(G, Q) \geq \frac{1}{\sqrt{2}}$$

for $k \leq 4$.

**PROOF.** This can be deduced from [11, page 250], where $m_{\chi}^{-1} |\chi(Q)| \leq (\sqrt{k} - 1)^{-1}$ is shown, if $k \geq 5$, and $m_{\chi}^{-1} |\chi(Q)| \leq 3/4$, if $k \leq 4$. \qed

Examples of such groups are $\operatorname{SL}(n,k)$ or $\operatorname{Sp}(n,k)$ with exception of $\operatorname{SL}(2,k) = \operatorname{Sp}(1,k)$ for $k = 2, 3$ and $\operatorname{Sp}(2,2)$. 

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Since the upper bound in the preceding proposition might be trivial the following determines another upper bound of the Kazhdan constant of a representation.

**Proposition 2.5** Let $Q$ be a conjugacy class of the compact group $G$, $q \in Q$ a representative, $U$ the closure of the subgroup generated by $q$, and $\pi$ a representation of $G$ on $H_\pi$, then

$$K(\pi, G, Q) \leq \sup_{\chi < \pi|_U} |\chi(q) - 1|,$$

where $\chi < \pi|_U$ denotes the irreducible characters of $U$ contained in $\pi|_U$.

**PROOF.** Let $\xi \in H_\pi^1$, then considering the whole unit sphere $H_\pi^1$ instead of $\pi(G)\xi$ gives

$$\sup_{g \in Q} \|\pi(g)\xi - \xi\| \leq \sup_{\eta \in H_\pi^1} \|\pi(q)\eta - \eta\| = \sup_{\chi < \pi|_U} |\chi(q) - 1|.$$

Hence $K(\pi, G, Q) \leq \sup_{\chi < \pi|_U} |\chi(q) - 1|$. □

3 Kazhdan constants for generating subsets

In this section arbitrary generating compact subsets are considered.

For subsets $Q', Q$ of a group $G$, let $Q'Q$ denote the set of all products $q'q$ of elements $q' \in Q'$ and $q \in Q$. Define $Q^1 = Q$, $Q^{n+1} = Q^nQ$ for positive integers $n$. Also let $Q^{-1}$ be the set of all $q^{-1}$ for $q \in Q$.

The first three easy results stated here seem to be not explicitly in the existing literature but for example a similar argument as in the proof of Proposition 3.2 was also used in [14, page 498]. They will then be used to give lower bounds for finite non-abelian simple groups which seem to be new.

**Lemma 3.1** Let $Q$ be a measurable generating set of a locally compact group $G$ with $Q^{-1} = Q$ and $1 \in Q$, then for every compact subset $C$ there is a positive integer $\nu$ such that $Q^\nu \supseteq C$.

**PROOF.** Let $\mu$ be the Haar measure of $G$. As $Q$ generates $G$, there is a positive integer $n$ such that $\mu(Q^n) > 0$. Now $Q^{2n}$ contains an open neighborhood $U$ of the identity, see for example [8, page 235]. The sets $Q^jU$ for $j = 1, 2, \ldots$
are an open covering of $C$. By compactness finitely many cover $C$ and hence there is a $k$ such that $C \subset Q^k U \subset Q^{2n+k}$. □

For a locally compact group $G$, a measurable generating subset $Q \subset G$, and a compact subset $C \subset G$ denote by $\nu(C, Q) = \nu$ the smallest positive integer such that $C \subset (Q^{-1} \cup \{1\} \cup Q)^{\nu}$. In the case $G$ is compact $\nu(G, Q)$ will be denoted by $\nu(Q)$.

The lower bounds of the Kazhdan constants computed in this section are dependent on $\nu(Q)$. The result will be applied to alternating groups and $\text{PSL}(n, k)$. For finite non-abelian simple groups a bound for the Kazhdan constant not depending on the nontrivial conjugacy class will be shown.

For an arbitrary compact subset $Q$ generating a locally compact group $G$ the integer $\nu$ in Lemma 3.1 may be used to determine a lower bound for the Kazhdan constant $K(G, Q)$.

**Proposition 3.2** Let $G$ be a locally compact group, $C$ and $Q$ arbitrary compact subsets of $G$ such that $Q$ generates $G$, and $\nu = \nu(C, Q)$, then $K(G, Q) \geq \nu^{-1} K(G, C)$.

**PROOF.** Let $\pi$ be a representation of $G$ on $H_\pi$ without nonzero fixed vector. For every $g \in C$ there are $q_1, \ldots, q_\nu \in Q^{-1} \cup \{1\} \cup Q$ such that $g = q_1 \cdots q_\nu$. Now using the triangle inequality for the associated “telescope sum”

$$\|\pi(g) \xi - \xi\| \leq \nu \sup_{q \in Q} \|\pi(q) \xi - \xi\|$$

for $\xi \in H_\pi$. Hence

$$\nu^{-1} K(\pi, G, C) \leq K(\pi, G, Q).$$

□

The following yields a lower bound for the Kazhdan constant of a compact group.

**Corollary 3.3** Let $G$ be a compact group, $Q$ an arbitrary compact generating set, and $\nu = \nu(Q)$, then $K(G, Q) \geq \sqrt{2} \nu^{-1}$. If $n = |G|$ is finite, $K(G, Q) \geq \sqrt{\frac{2n}{n-1}} \nu^{-1}$.

**PROOF.** This can be deduced from $K(G, G) = \sqrt{2}$ for an infinite compact group and $K(G, G) = \sqrt{\frac{2n}{n-1}}$ for a finite group, see [7], and Proposition 3.2. □
An application of the preceding corollary and Theorem 1.1 shows
\[
\frac{\text{Re} \chi(Q)}{m} \leq 1 - \frac{1}{(\nu(Q))^2}
\]
for every nontrivial irreducible character \( \chi \) of degree \( m = \chi(1) \).

If \( G \) is non-abelian, finite, and simple, then every nontrivial conjugacy class generates \( G \). For those groups, the smallest positive integer \( n \) such that \( Q^n = G \) for all nontrivial conjugacy classes \( Q \) is called the covering number and denoted by \( \text{cn}(G) \). Obviously \( \nu(Q) \leq \text{cn}(G) \) for every nontrivial conjugacy class \( Q \subset G \).

Calculations of \( \text{cn}(G) \) for various groups yield the following estimates:

**Theorem 3.4** (1) For the alternating group \( A_n, n \geq 5 \), the Kazhdan constant \( K(A_n, Q) > 2\sqrt{2}/\max\{6, n\} \) for every conjugacy class \( Q \neq \{1\} \).

(2) Let \( G = \text{PSL}(n, k) \), the projective special linear group of a vector space of dimension \( n \geq 2 \) over a finite field with \( k \geq 4 \) elements, then \( K(G, Q) > \sqrt{2}/\max\{3, n\} \) for every conjugacy class \( Q \neq \{1\} \).

**Proof.**

(1) For the alternating group \( A_n \) on \( n \geq 6 \) letters \( \text{cn}(A_n) \) is shown in [1] to be the integer part of \( \frac{n}{2} \). The case \( n = 5 \) is contained in [15] showing \( \text{cn}(A_5) = 3 \). This proves 1.

(2) The result for \( G = \text{PSL}(n, k) \) can be deduced as follows. The reference [1, page 240] shows \( \text{cn}(G) = 3 \), if \( n = 2 \) and \( k \geq 4 \). In [17] it is shown that \( \text{cn}(G) = n \), provided \( n \geq 3 \) and \( k \geq 4 \). The combination of the results yields \( \text{cn}(G) = \max\{3, n\} \). \( \square \)

For a finite non-abelian simple group there is also the following general lower bound for the Kazhdan constant which can be derived from [25]. For large enough groups [18, page 385] would provide a better upper bound for the covering number but it involves a constant which is only known to exist. In the proof of this result also a character value estimate for finite groups of Lie type is involved but it is only derived from [11] and would give no improvement to Theorem 2.4.

**Theorem 3.5** Let \( G \) be a finite non-abelian simple group, \( n = |G| \), and \( Q \) a nontrivial conjugacy class, then
\[
K(G, Q) \geq \frac{(2 - \sqrt{3}) \sqrt{2n}}{(n - 1) \ln(n - 1)}.
\]
PROOF. This can be deduced from Corollary 3.3 and [25, page 538] which shows \( cn (G) \leq (2 + \sqrt{3}) \sqrt{n - 1} \ln (n - 1). \) □

The next proposition shows that the Kazhdan constant is 0, if the compact subset generates a non-dense subgroup in \( G \). The existence of positive Kazhdan constants for finite subsets of compact groups is shown in [26, page 3409]. Considering the following proposition which seems to be new these must generate a dense subgroup.

**Proposition 3.6** Let \( G \) be a compact group and \( Q \) a compact subset of \( G \) such that the subgroup generated by \( Q \) is not dense in \( G \), then \( K (G, Q) = 0. \)

**PROOF.** The compact subset \( Q \) is contained in a closed subgroup \( U \neq G \). Let \( \pi \) be the representation of \( G \) induced by the trivial representation of \( U \). It is equivalent to the quasi-regular representation on \( L^2 (G/U) \). Since \( U \neq G \), this representation is not trivial. Hence \( \pi \) contains an irreducible representation \( \sigma \) different from the trivial representation. By Frobenius reciprocity, see for example [10, pages 160–162], the restriction \( \sigma|_U \) contains the trivial representation. Hence \( K (G, Q) = 0. \) □

This is false in general: Consider \( G = \text{SL} (n, \mathbb{R}) \), \( n \geq 3 \), and the set \( Q \) of matrices

\[
\begin{pmatrix}
1 & 2 \\
0 & 1
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
1 & 0 \\
2 & 1
\end{pmatrix}
\]

embedded in the upper left corner of \( G \). Clearly, \( Q \) generates a proper closed subgroup of \( G \), but the Kazhdan constant \( K (G, Q) > 0 \), see [27, page 835].

Proposition 3.6 shows the expression in Theorem 1.1 is 0, if \( Q \) is a conjugacy class which generates no dense subgroup. The converse is not true. Consider for example the circle \( T \). An element \( q = e^{2\pi i t} \) with \( t \in \mathbb{R} \setminus \mathbb{Q} \) generates a dense subgroup of \( T \). The set of equivalence classes of irreducible representations of \( T \) is \( \hat{T} = \{ \chi_n : n \in \mathbb{Z} \} \) with \( \chi_n (c) = c^n \) for \( c \in T \). This expression can be made arbitrarily close to 1 so \( \inf_{n \neq 0} \sqrt{2 - 2 \Re \chi_n (q)} = 0. \) But there is a converse if \( Q \) generates \( G \) algebraically. This is the general fact \( K (G, Q) > 0 \) for such a set.

4 The symmetric group

In this section Theorem 1.2 is proven which determines a lower bound for the symmetric group \( S_n \) on \( n \) letters. This is done by transferring bounds for
subgroups to bounds for the group considering in addition those irreducible characters of the group, whose restriction to the subgroup contains the trivial character.

Normalized characters of the symmetric group have also been studied for example in [9] whose result we will use here to prove the general case and in [25] and [21] but with explicit estimates in the last-mentioned only for conjugacy classes of large cycles which cannot be applied in the case of 2-cycles investigated here.

The following determines an upper bound for the real part of the normalized characters of a compact group $G$ from upper bounds of a closed subgroup $U$ which has non-empty intersection with the conjugacy class $Q$.

**Proposition 4.1** Let $G$ be a compact group, $Q$ a conjugacy class, and $U$ a closed subgroup of $G$ such that $Q \cap U \neq \emptyset$, then $Q \cap U = \bigcup_{t \in T} V_t$ a union of conjugacy classes $V_t$ of $U$. Let $Z$ be a set of irreducible characters of $G$, $Z'$ a set of irreducible characters of $U$, and $E \subset Z$ the set of all characters $\chi \in Z$ such that the restriction $\chi|_U$ contains an irreducible character not in $Z'$, then

$$\sup_{\chi \in Z} m_{\chi}^{-1} \text{Re} \, \chi (Q) \leq \max \left\{ \sup_{\chi \in E} m_{\chi}^{-1} \text{Re} \, \chi (Q), \inf_{t \in T} \sup_{\chi \in Z'} m_{\chi}^{-1} \text{Re} \, \chi (V_t) \right\},$$

where $m_{\chi} = \chi(1)$ is the degree of $\chi$.

**PROOF.** Let $\chi \in Z \setminus E$, then there are $\psi_1, \ldots, \psi_n \in Z'$ such that $\chi|_U = \sum_{s=1}^n \psi_s$. So

$$m_{\chi}^{-1} \text{Re} \, \chi (u) \leq \sup_{\psi \in Z'} m_{\psi}^{-1} \text{Re} \, \psi (u)$$

follows by considering it as a convex combination and bounding it by the supremum. In particular this holds for $u \in V_t$ and hence

$$m_{\chi}^{-1} \text{Re} \, \chi (Q) \leq \inf_{t \in T} \sup_{\psi \in Z'} m_{\psi}^{-1} \text{Re} \, \psi (V_t).$$

Using the above therefore yields

$$\sup_{\chi \in Z} m_{\chi}^{-1} \text{Re} \, \chi (Q) \leq \max \left\{ \sup_{\chi \in E} m_{\chi}^{-1} \text{Re} \, \chi (Q), \inf_{t \in T} \sup_{\psi \in Z'} m_{\psi}^{-1} \text{Re} \, \psi (V_t) \right\}.$$
The result is now used to find the Kazhdan constants of the symmetric groups relative to the conjugacy class of all 2-cycles.

For the following see for example [28, pages 95–120]. The equivalence classes of irreducible unitary representations of $S_n$ can be parametrized by Young frames. A Young frame on $n$ is a sequence $b_1 \geq b_2 \geq \ldots \geq b_r \geq 1$ of integers such that $\sum_{k=1}^r b_k = n$ and is denoted by $(b_1, b_2, \ldots, b_r)$. The group $S_{n-1}$ is considered as a subgroup of $S_n$ in the natural way. If $F = (b_1, \ldots, b_r)$ and $F' = (b'_1, \ldots, b'_r)$ are Young frames on $n$ respectively $n-1$, then the notation $F' \triangleleft F$ means $b'_s \leq b_s$ for $1 \leq s \leq r'$.

**Proposition 4.2** The restriction of a representation $\pi_F$ of $S_n$ to $S_{n-1}$ is

$$\pi_F|_{S_{n-1}} \cong \bigoplus_{F' \triangleleft F} \pi_{F'}.$$ 

The trivial representation of $S_n$ corresponds to the frame $F = (n)$. This shows the trivial representation is contained in $\pi_F|_{S_{n-1}}$ if and only if $(n-1) = F' \triangleleft F$. But this can only be the case, when $F = (n)$ or $F = (n-1, 1)$.

The conjugacy classes of $S_n$ are determined by the cycle structure. This means a conjugacy class consists of all permutations, whose number of 1-cycles is $k_1$, the number of 2-cycles is $k_2$, and so on until the number of $n$-cycles $k_n$, if written as disjoint cycles. Then $\sum_{j=1}^n jk_j = n$ and the conjugacy class will be denoted by $1^{k_1}2^{k_2}\cdots n^{k_n}$.

Propositions 4.1 and 4.2 yield the following.

**Corollary 4.3** Let $n \geq 3$, $Z$ resp. $Z'$ be the set of all nontrivial characters of $S_n$ resp. $S_{n-1}$, and $Q = 1^{k_1}2^{k_2}\cdots (n-1)^{k_{n-1}}$ with $k_1 \geq 1$, then $V = S_{n-1} \cap Q = 1^{k_1-1}2^{k_2}\cdots (n-1)^{k_{n-1}}$ and

$$\sup_{\chi \in Z} m^{-1} \chi(Q) \leq \max \left\{ m^{-1} \chi_{(n-1,1)}(Q), \sup_{\chi \in Z'} m^{-1} \chi(Q) \right\},$$

where $\chi_{(n-1,1)} = \tr \pi_{(n-1,1)}$.

For the frame $F = (n-1, 1)$ the following holds.

**Lemma 4.4** The values of the character corresponding to the Young frame $F = (n-1, 1)$ are

$$\chi_{(n-1,1)}\left(1^{k_1}2^{k_2}\cdots n^{k_n}\right) = k_1 - 1.$$ 

Now Theorem 1.2 can be proven.
Proof of Theorem 1.2. The general result for arbitrary $Q$ with $\text{sgn} \, Q = -1$ can be deduced from [9, page 167] stating $m^{-1}_\chi |\chi (Q)| \leq \frac{n-3}{n-1}$ for every $\chi$ with $m_\chi \geq 2$ and every conjugacy class $Q \neq \{1\}$.

The proof showing equality in the case $Q = 1^{n-2}2^1$ can be done independently of this result by the results developed in this section as follows.

Let $n = 2$, then $S_2 \cong C_2$ and $K \, (S_2, Q) = 2$ by the example in Section 2. Now induction, Lemma 4.4, and computing the maximum in Corollary 4.3 yields

$$K \, (S_n, Q) \geq \frac{2}{\sqrt{n-1}}.$$ 

Considering the representation $\pi_{(n-1,1)} \otimes \text{id}_{H^{(n-1,1)}}$ and a similar argument as in the proof of Proposition 2.2 yields

$$K \, (S_n, Q) \leq \sqrt{2 - \frac{2}{n-1}} \, \text{Re} \chi_{(n-1,1)} (Q) = \frac{2}{\sqrt{n-1}}.$$ 

So equality holds. \(\square\)

5 The special unitary group

In this section we consider $G = \text{SU} (n)$. In the first part $n = 2$. Let $Q_t$ be the set of all $g \in G$ with eigenvalues $e^{\pm it}$ and $q_t = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$, then $Q_t = \{ h^{-1} q_t h : h \in G \}$. For every nonnegative integer $m$ there is exactly one equivalence class of irreducible representations of degree $m + 1$, see for example [10, page 143]. An orthonormal basis of the space $H_m$ of such a representation $\pi_m$ consists of the homogeneous polynomials $f_0, \ldots, f_m$ with $f_r (z, w) = \sqrt{\frac{(m+1)!}{r!(m-r)!}} z^r w^{m-r}$ for $r = 0, \ldots, m$ and $\pi_m \, (q_t)$ acts diagonally with respect to this basis, as

$$\pi_m \, (q_t) \, f_r \, (z, w) = f_r \left( e^{-it} z, e^{it} w \right) = e^{i(m-2r)t} \, f_r \, (z, w)$$

by definition of $\pi_m$, see for example [10, page 143].

The following is a consequence of Theorem 1.1.
**Theorem 5.1** Let $G = SU(2)$ and $Q_t$ the conjugacy class of $q_t = \left( \begin{array}{cc} e^{it} & 0 \\ 0 & e^{-it} \end{array} \right)$.

Then

$$K(G, Q_t) = \inf_{m \geq 1} \sqrt{2 - \frac{2}{m + 1} \frac{\sin((m + 1)t)}{\sin t}} \geq |\sin t|.$$ 

**PROOF.** Let $\pi_m$ be the irreducible representation of $G$ on $H_m$ of degree $m + 1$. The value of the character $\chi_m = \text{tr} \pi_m$ for $q_t$ is a geometric sum $\chi_m(q_t) = \frac{\sin((m+1)t)}{\sin t}$. Then Theorem 1.1 shows the equality.

The inequality is obtained by estimating $\inf_{m \geq 1} \sqrt{2 - \frac{2}{m + 1} \frac{\sin((m+1)t)}{\sin t}}$. The value of this expression will be estimated for $t$ in three different intervals depending on $m$.

An elementary calculation shows

$$2 (m + 1) - 2 \text{Re} \chi_m(q_t) = 4 \sum_{r=0}^{m} (\sin((m/2 - r)t))^2.$$ 

For $0 \leq t \leq \frac{\pi}{2}$ the function $t \mapsto (\sin t)^2$ is increasing and so all functions $t \mapsto (\sin((m/2 - r)t))^2$ are increasing for $0 \leq t \leq \frac{\pi}{m}$. Hence

$$2 - \frac{2}{m + 1} \text{Re} \chi(q_t) \geq 4 (\sin(t/2))^2 \geq (\sin t)^2.$$ 

For $\arcsin(2/(m + 1)) \leq t \leq \pi - \arcsin(2/(m + 1))$

$$2 - \frac{2}{m + 1} \frac{\sin((m + 1)t)}{\sin t} \geq 1 \geq (\sin t)^2.$$ 

As $\left| \frac{\sin((m+1)(\pi-t))}{\sin(\pi-t)} \right| = \frac{\sin((m+1)t)}{\sin t}$ for $0 \leq t \leq \frac{\pi}{m+1}$, by symmetry

$$2 - \frac{2}{m + 1} \frac{\sin((m + 1)t)}{\sin t} \geq (\sin t)^2$$

for $\pi - \frac{\pi}{m+1} \leq t \leq \pi$.

As $\arcsin(2/(m + 1)) \leq \frac{\pi}{m+1} \leq \frac{\pi}{m}$,

$$K(G, Q_t) \geq \sin t$$
for $0 \leq t \leq \pi$. □

The expression of the previous Kazhdan constant can be simplified. Namely the infimum is in fact a minimum of only finitely many terms, i.e. there exists $M \in \mathbb{N}$ such that

$$\inf_{m \geq 1} \sqrt{2 - \frac{2}{m+1} \sin \left( \frac{(m+1)t}{\sin t} \right)} = \min_{m=1,\ldots,M} \sqrt{2 - \frac{2}{m+1} \sin \left( \frac{(m+1)t}{\sin t} \right)}.$$ 

This is shown by the following.

**Lemma 5.2** It holds that

$$\sup_{m \geq 1} \frac{\sin \left( \frac{(m+1)t}{m+1} \right)}{m+1} = \max_{m=1,\ldots,10} \frac{\sin \left( \frac{(m+1)t}{m+1} \right)}{m+1}.$$ 

**PROOF.** By convexity of $\sin$ in the interval $[0, \pi]$ holds

$$\frac{\sin \left( \frac{rt}{t} \right)}{r} \leq \frac{\sin (2t)}{2}$$

for $t \in [0, \pi/r]$. The boundedness of $\sin$ shows

$$\frac{\sin \left( \frac{rt}{t} \right)}{r} \leq \frac{\sin (2t)}{2}$$

for $t \in [\arcsin (2/r), \pi/4]$ as $x \mapsto \sin x$ is increasing in the interval $[0, \pi/2]$. As $2x/\pi \leq \sin x$ for $x \in [0, \pi/2]$ the inequality is proven for $t \in [0, \pi/4]$.

Similar arguments for $r \geq 3$ show

$$\frac{\sin \left( \frac{rt}{t} \right)}{r} \leq \frac{\sin (3t)}{3}$$

for $t \in [\pi - \pi/r, \pi]$ and $t \in [5\pi/6, \pi - \arcsin (3/r)/3]$. Hence the inequality is proven for $t \in [5\pi/6, \pi]$.

For $t \in [\pi/6, 5\pi/6]$ holds $\sin t > 1/2$. So $\sin (rt) > 1/2$ for $rt \in 2s\pi + [\pi/6, 5\pi/6]$ and $0 \leq s \leq (r-1)/2$. For $r = 2, 3, 4, 5, 6$ there are the intervals

\[
\begin{align*}
\left[ \frac{\pi}{12}, \frac{5\pi}{12} \right], & \quad \left[ \frac{13\pi}{18}, \frac{17\pi}{18} \right], & \quad \left[ \frac{13\pi}{24}, \frac{17\pi}{24} \right], \\
\left[ \frac{13\pi}{18}, \frac{17\pi}{18} \right], & \quad \left[ \frac{13\pi}{24}, \frac{17\pi}{24} \right], & \quad \left[ \frac{13\pi}{24}, \frac{17\pi}{24} \right]
\end{align*}
\]
which cover \([\pi/4, 5\pi/6]\). This means for \(t\) in this interval

\[
\sup_{m \geq 1} \frac{\sin ((m + 1)t)}{m + 1} > \frac{1}{12},
\]

i.e. \(m \geq 11\) does not need to be considered. \(\square\)

A more detailed analysis can show that \(M = 4\) is enough.

For \(G = U(2)\), the group of all unitary \(2 \times 2\) matrices, the irreducible representations \(\rho_{m,n}\) of \(U(2)\) with \(m \geq 0\), \(n \in \mathbb{Z}\), and \(m \equiv n \mod 2\) are representatives of the equivalence classes of irreducible representations of \(U(2)\), see again [10, page 147]. They are defined by \(\rho_{m,n}(e^{it}h) = e^{int} \pi_m(h)\) for \(h \in SU(2)\). The following is a consequence of Theorem 5.1.

**Corollary 5.3** Let \(G = U(2)\) and \(Q_t\) the conjugacy class of \(q_t = \begin{pmatrix} e^{it_1} & 0 \\ 0 & e^{it_2} \end{pmatrix}\).

Then

\[
K(G, Q_t) \geq |\sin ((t_1 - t_2)/2)|.
\]

**PROOF.** As \(q_t = e^{(t_1 + t_2)/2} \begin{pmatrix} e^{(t_1 - t_2)/2} & 0 \\ 0 & e^{(t_2 - t_1)/2} \end{pmatrix}\), the proof of Theorem 5.1 implies \(\text{tr} \rho_{m,n}(g) = e^{in(t_1 + t_2)/2} \frac{\sin((m+1)(t_1-t_2)/2)}{\sin((t_1-t_2)/2)}\) for \(g \in Q_t\). Now

\[
\sqrt{2 - \frac{2}{m+1} \cos(n(t_1 + t_2)/2)} \frac{\sin((m+1)(t_1-t_2)/2)}{\sin t} \geq |\sin ((t_1 - t_2)/2)|
\]

follows from the proof of the last theorem and hence

\[
K(G, Q_t) \geq |\sin ((t_1 - t_2)/2)|. \quad \square
\]

Here it is possible to compute the exact value of the Kazhdan constant for the representation \(\pi_2\) of \(SU(2)\) and this shows equality does not hold in general in Proposition 2.2 for the lower bound of the Kazhdan constant \(K(\pi, G, Q)\) of a representation \(\pi\), as is the case for \(K(G, Q)\).
Theorem 5.4 For the representation \( \pi = \pi_2 \) of \( G = SU(2) \)

\[
K(\pi, G, Q_t) = 2|\sin t|.
\]

PROOF. Let \( P \) be the orthogonal projection onto the space spanned by \( f_1 \). Then

\[
(K(\pi, G, Q_t))^2 = \inf_{\xi \in H^1} \sup_{h \in G} \| \pi(q_t) \pi(h) \xi - \pi(h) \xi \| ^2
= |e^{2it} - 1|^2 \left( 1 - \sup_{\xi \in H^1} \inf_{h \in G} \| P \pi(h) \xi \| ^2 \right),
\]

as \( \pi(q_t) \) is diagonal with respect to the chosen basis and has the eigenvalues 1 and \( e^{\pm 2it} \). It remains to show \( \inf_{h \in G} \| P \pi(h) \xi \| = 0 \) which then implies \( K(\pi, G, Q_t) = 2|\sin t| \).

So let \( h = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2) \), then

\[
P \pi(h) \xi = (-\sqrt{2}ab\xi_0 + (|a|^2 - |b|^2) \xi_1 + \sqrt{2}ab\xi_2) f_1,
\]

where \( \xi = \xi_0 f_0 + \xi_1 f_1 + \xi_2 f_2 \). If \( \xi_1 = 0 \), let \( a = 1, b = 0 \), then \( P \pi(h) \xi = 0 \).

Now assume \( \xi_1 \neq 0 \), without loss of generality \( \xi_1 \in \mathbb{R} \). Hence

\[
\text{Im} \left( -\sqrt{2}ab\xi_0 + (|a|^2 - |b|^2) \xi_1 + \sqrt{2}ab\xi_2 \right) = -\sqrt{2} \text{Im} \left( ab \left( \xi_0 + \xi_2 \right) \right).
\]

If \( |a| \) and \( |b| \) are known, the above vanishes, if \( ab = |ab| e^{-i\vartheta} \), where \( \vartheta \) is determined by \( \xi_0 + \xi_2 = e^{i\vartheta} |\xi_0 + \xi_2| \). This can be achieved by letting \( a \in \mathbb{R} \) and choosing the argument of \( b \) accordingly. The values of \( |a| \) and \( |b| \) are determined by considering

\[
0 = \sqrt{2} \text{Re} \left( -ab\xi_0 + ab\xi_2 \right) + (|a|^2 - |b|^2) \xi_1
= -\sqrt{2} |ab| \text{Re} \left( e^{-i\vartheta} \left( \xi_0 - \xi_2 \right) \right) + (|a|^2 - |b|^2) \xi_1
\]

which can be solved by suitably chosen \( |a| \) and \( |b| \). \( \square \)

So \( K(\pi_2, G, Q_t) > \sqrt{2 - \frac{2}{3} \sin(3t)} \) for \( 0 < t < \pi \), as by an elementary calculation using the addition formula for \( \sin \)

\[
2 - \frac{2}{3} \sin(3t) = \frac{8}{3} (\sin t)^2 < 4 (\sin t)^2.
\]

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The aim of the following is to develop recursion formulas for normalized characters of \( SU(n) \) which allow a proof of Theorem 1.3. Related works are for example [23] and [24]. But neither are applicable here. In [23] the normalized character values are integrated over some subgroup and estimated only afterwards and in [24] normalized character values are considered only for reflections which are elements conjugate to a diagonal matrix with only one entry different from 1. But here we aiming for estimates for all group elements.

Let \( \mu = (\mu_1, \ldots, \mu_n) \) a tuple of integers \( \mu_s \) with \( \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \), \( \chi_\mu(x) = \det M_\mu(x) / \det M_0(x) \), where \( M_\mu(x) \) is the matrix with the coefficients \( x_r^{\mu_r+n-s} \) for \( r, s = 1, \ldots, n \) and \( x \in T^n \). The matrix \( M_0(x) \) is a Vandermonde matrix and so \( \det M_0(x) = \prod_{1 \leq r < s \leq n} (x_r - x_s) \).

The equivalence classes of irreducible representations of \( U(n) \) can be parametrized by these tuples with \( \mu_n \geq 0 \) and the value of the character of \( g \in U(n) \) corresponding to \( \mu \) is \( \chi_\mu(x) \), where \( x_1, \ldots, x_n \) are the eigenvalues of \( g \). The representations corresponding to \( SU(n) \) are exactly those with \( \mu_n = 0 \).

For \( U(n-1) \) considered as a subgroup of \( U(n) \) the following branching relation

\[
\pi_\mu|_{U(n-1)} \cong \bigoplus_{\kappa_1=\mu_2=\mu_3}^{\mu_1} \bigoplus_{\kappa_2=\mu_1=\mu_3}^{\mu_2} \cdots \bigoplus_{\kappa_n-2=\mu_n-1}^{\mu_n-2} \bigoplus_{\kappa_n-1=\mu_n}^{\mu_n-1} \pi_\kappa
\]

holds. This implies for the characters

\[
\chi_\mu(x) = \sum_{\kappa_1=\mu_2=\mu_3}^{\mu_1} \sum_{\kappa_2=\mu_1=\mu_3}^{\mu_2} \cdots \sum_{\kappa_n-2=\mu_n-1}^{\mu_n-2} \sum_{\kappa_n-1=\mu_n}^{\mu_n-1} \chi_\kappa(y),
\]

where \( x = (x_1, \ldots, x_{n-1}, 1) \in T^n \) and \( y = (x_1, \ldots, x_{n-1}) \). As the representations are irreducible for arbitrary \( x \in T^n \) holds

\[
|\chi_\mu(x)| \leq \sum_{\kappa_1=\mu_2=\mu_3}^{\mu_1} \sum_{\kappa_2=\mu_1=\mu_3}^{\mu_2} \cdots \sum_{\kappa_n-2=\mu_n-1}^{\mu_n-2} \sum_{\kappa_n-1=\mu_n}^{\mu_n-1} |\chi_\kappa(y)|.
\]

This yields a recursion formula of \( \chi_\mu(x) \) for \( x = (1, \ldots, 1) \). In particular the value for \( x = (1, \ldots, 1) \) is

\[
m_\mu = \sum_{\kappa_1=\mu_2=\mu_3}^{\mu_1} \sum_{\kappa_2=\mu_1=\mu_3}^{\mu_2} \cdots \sum_{\kappa_n-2=\mu_n-1}^{\mu_n-2} \sum_{\kappa_n-1=\mu_n}^{\mu_n-1} m_\kappa.
\]

Let in the following

\[
N_n = \{ \mu \in \mathbb{Z}^n : \mu_1 \geq 1, \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n-1 \geq \mu_n = 0 \},
\]
then holds \( n \leq \inf_{\mu \in N_n} m_\mu \). In fact, \( 2 = \inf_{\mu \in N_2} m_\mu \) and for \( n \geq 3 \) by induction

\[
\inf_{\mu \in N_n} m_\mu \geq 1 + \inf_{\mu \in N_n} \left((\mu_1 - \mu_2 + 1) \cdots (\mu_{n-1} - \mu_n + 1) - 1\right) \inf_{\kappa \in N_{n-1}} m_\kappa \\
\geq n,
\]
as \( m_{\kappa'} = m_\kappa \) for \( \kappa' = (\kappa_1 - \kappa_{n-1}, \ldots, \kappa_{n-2} - \kappa_{n-1}, 0) \). Actually \(|\chi_{\kappa'}(y)| = |\chi_\kappa(y)|\) for \( y = (x_1, \ldots, x_{n-1}) \in T^{n-1}\).

**Proposition 5.5** Suppose \( x = (x_1, \ldots, x_n) \in T^n \) and \( y = (x_1, \ldots, x_{n-1}) \) where \( n \geq 3 \), then

\[
\sup_{\mu \in N_n} m_\mu^{-1} |\chi_\mu(x)| \leq \frac{1}{n} + \frac{n-1}{n} \sup_{\kappa \in N_{n-1}} m_\kappa^{-1} |\chi_\kappa(y)|.
\]

**PROOF.** If \( \mu_2 = \mu_{n-1} \), then using the estimates above

\[
m_\mu^{-1} |\chi_\mu(x)| \leq \frac{1}{m_\mu} + \frac{m_\mu - 1}{m_\mu} \sum_{\kappa_1 = \mu_2}^{\mu_1} \cdots \sum_{\kappa_{n-1} = 0}^{\min\{\mu_{n-1}, \kappa_{n-1} - 1\}} \frac{m_\kappa}{m_\mu - 1} m_\kappa^{-1} |\chi_\kappa(y)|
\]

\[
\leq \frac{1}{n} + \frac{n-1}{n} \sup_{\kappa \in N_{n-1}} m_\kappa^{-1} |\chi_\kappa(y)|,
\]
as \( n \leq \inf_{\mu \in N_{n-1}} m_\mu \). If \( \mu_2 > \mu_{n-1} \), again with the estimates above

\[
m_\mu^{-1} |\chi_\mu(x)| \leq \sup_{\kappa \in N_{n-1}} m_\kappa^{-1} |\chi_\kappa(y)|. \quad \square
\]

**Corollary 5.6** Let \( n \geq 2 \) and \( x_r = e^{it_r} \) for \( r = 1, \ldots, n \), then

\[
\inf_{\mu \in N_n} \sqrt{2 - \frac{2}{m_\mu}} |\chi_\mu(x)| \geq \sqrt{\frac{2}{n}} \left|\sin\left(\frac{(t_1 - t_2)}{2}\right)\right|.
\]

**PROOF.** For \( n = 2 \), this follows from Corollary 5.3.

Let \( n \geq 3 \), then Proposition 5.5 yields

\[
\inf_{\mu \in N_n} \sqrt{2 - \frac{2}{m_\mu}} |\chi_\mu(x)| \geq \sqrt{\frac{n-1}{n}} \inf_{\kappa \in N_{n-1}} \sqrt{2 - \frac{2}{m_\kappa}} |\chi_\kappa(y)|.
\]

Induction together with Corollary 5.3 shows

\[
\inf_{\mu \in N_n} \sqrt{2 - \frac{2}{m_\mu}} |\chi_\mu(x)| \geq \sqrt{\frac{2}{n}} \left|\sin\left(\frac{(t_1 - t_2)}{2}\right)\right|. \quad \square
\]
Now it is possible to prove Theorem 1.3.

**Proof of Theorem 1.3.** Let $x_r = e^{it_r}$ for $r = 1, \ldots, n$, then, by the corollary above,

$$K(G, Q_t) \geq \sqrt{\frac{2}{n}} \sin \left(\frac{(t_1 - t_2)/2}{n}\right).$$

As permutation of the eigenvalues $x_1, \ldots, x_n$ does not change $Q_t$,

$$K(G, Q_t) \geq \sqrt{\frac{2}{n}} \max_{1 \leq r, s \leq n} \left| \sin \left(\frac{(t_r - t_s)/2}{n}\right) \right|. \quad \Box$$

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**References**


