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Neighbourhood Graphs of Cayley Graphs for Finitely-Generated Groups

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Abstract

In this short note the neighbourhood graph of a Cayley graph is considered. It has as nodes a symmetric generating set of a finitely-generated group Γ. Two nodes are connected by an edge if one is obtained from the other by multiplication on the right by one of the generators. Two necessary conditions on the graphs are shown. One is a condition on the degrees of the graph, the other concerns complete subgraphs.

1. Introduction

Let Γ be a finitely-generated group. A *symmetric generating set* of Γ is a subset S of Γ with $1 \notin S$ and $s^{-1} \in S$ for all $s \in S$ such that S generates the group Γ.

The *Cayley graph* of Γ with respect to S is the graph which has as vertices the elements of Γ and as edges $(\gamma, \gamma s)$ where $\gamma \in \Gamma$ and $s \in S$, see for example [1] or [3].

The *induced subgraph* of a graph on a subset of the vertices is the subgraph with all edges of the graph with both endpoints in the subset.

The *neighbourhood graph* of a vertex is the induced subgraph on the neighbours of this vertex. For graphs with given neighbourhoods see [4].

Let now S be a finite symmetric generating set of a finitely-generated group Γ. The neighbourhood graph (S, K) of $1 \in \Gamma$ of the Cayley graph of Γ with respect to S with edges K will be investigated. One motivation for considering these graphs is provided by a theorem of Żuk stating that the group Γ has Kazhdan's property T if the graph is connected and the smallest positive eigenvalue λ of a certain Laplacian on the graph (S, K) is strictly greater than $1/2$, see [6].

Here the problem is considered of which finite connected graphs can occur in this way as the neighbourhood graph of a Cayley graph for some finitely-generated group. Two necessary conditions are given. The first is a condition on

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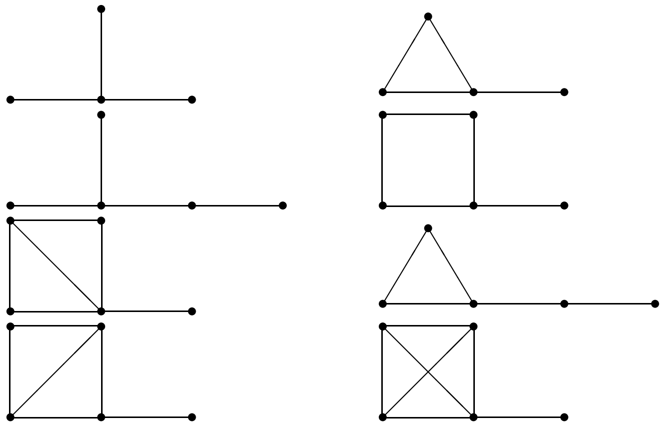
the degrees $N(s) = |\{t \in S : (s,t) \in K\}|$ of the graph. It says that the number of nodes with a given odd number of neighbours is always even.

Theorem 1.1: *If* $n \equiv 1 \bmod 2$ *then* $|N^{-1}(\{n\})| \equiv 0 \bmod 2$.

Note that this is certainly stronger than the general fact that $\sum_{s\in S} N(s) = |K| \equiv 0 \bmod 2$.

The theorem excludes for example all of the following graphs from consideration.



Graphs not fulfilling the degree condition for $n = 1$

The second condition sets up a correspondence between complete subgraphs and finite subgroups of Γ contained in $S \cup \{1\}$. Clearly a subgroup contained in $S \cup \{1\}$ induces a complete subgraph of (S, K). But also the following holds.

Theorem 1.2: *Let* S *be a finite symmetric generating set of* Γ, c_r *the number of complete subgraphs with* r *nodes, and* u_r *the number of subgroups of order* $r+1$ *of* Γ *contained in* $S \cup \{1\}$. *Then* $c_r = u_r + \sum_{j=1}^{m} p_j k_j$ *for some* $k_j \geq 0$ *where* $p_1, \ldots, p_m$ *are the distinct primes dividing* $r+1$.

The cases $r = 1$ or 2 yield the following observations: For $r = 1$ the theorem implies the obvious fact that the number u_1 of elements of order 2 is $\equiv c_1 \bmod 2$, where $c_1 = |S|$. When $r = 2$ the number of elements of order 3 is equal to $2u_2$, and $c_2 = |K|/2$, and the theorem implies $c_2 = u_2 + 3k_1$, hence we obtain the following.

Corollary 1.3: *If* n *is the number of elements of order* 3 *in* S, *then* $|K| \equiv n \bmod 6$.

This excludes for example the graph with 3 nodes and 2 edges because in that case $|K| = 4$ while $n \leq |S| = 3$. More examples like this are provided in Section 2.

The appendix contains a list of all graphs with ≤ 5 nodes and the associated groups and generating sets. Theorems 1.1 and 1.2 were used to exclude certain graphs which have no such description.

None of the infinite groups which appear in our list has Kazhdan's property T. So if there exists a generating set of an infinite finitely-generated group with Kazhdan's property T such that the theorem of Żuk can be applied it must have a set with ≥ 6 generators forming a connected graph.

The smallest known graph of a generating set of an infinite finitely-generated group for which the theorem of Żuk establishes property T has 14 nodes. It is associated with a so-called $\widetilde{A}_2$-group, see [2] and [6]. The graph of generators is the incidence graph of points and lines of a projective plane over a finite field with two elements.

It is a pleasure to thank M. B. Bekka, P. de la Harpe, H. Führ, and G. Schlichting for their comments and suggestions, and the referee for suggestions, remarks, and the reference [4]. Partial financial support from the Centre de Coopération Universitaire Franco-Bavarois is acknowledged.

2. Consequences of Theorem 1.2

Starting with a given graph, the subgroups contained in $S \cup \{1\}$ are not known if the graph is presumed to be associated with a group with a generating set S, however the last theorem gives lower bounds for the number of subgroups of order $r+1$ contained in $S \cup \{1\}$.

Corollary 2.1: *Let S be a generating set of a group Γ, c_r the number of complete subgraphs of (S, K) with r nodes, and let v_r be the smallest non-negative integer expressible in the form $c_r - \sum_{j=1}^{m} p_j k_j$ where $p_1, \ldots, p_m$ are the distinct primes dividing $r+1$, and $k_1, \ldots, k_m$ are non-negative integers. Then there are at least v_r subgroups of order $r+1$ of Γ contained in $S \cup \{1\}$.*

In this way the minimum possible number of subgroups is determined from the complete subgraphs which must be contained in $S \cup \{1\}$. If $U_1 \neq U_2$ are two subgroups contained in $S \cup \{1\}$ then $|U_1 \cap U_2|$ is a *proper* divisor of at least one of $|U_1|$ and $|U_2|$. Let $d(u, v) = \gcd(u, v)$ if $u \neq v$ and $d(u, v) = u/p$ where p is the smallest prime dividing u if $u = v$. Then $|U_1 \cap U_2| \leq d(|U_1|, |U_2|)$.

Proposition 2.2: *Let S be a symmetric generating set of a group Γ such that $S \cup \{1\}$ contains pairwise distinct finite subgroups $U_1, \ldots, U_n$ of Γ of order $|U_k| \geq 2$ for all $1 \leq k \leq n$, then*

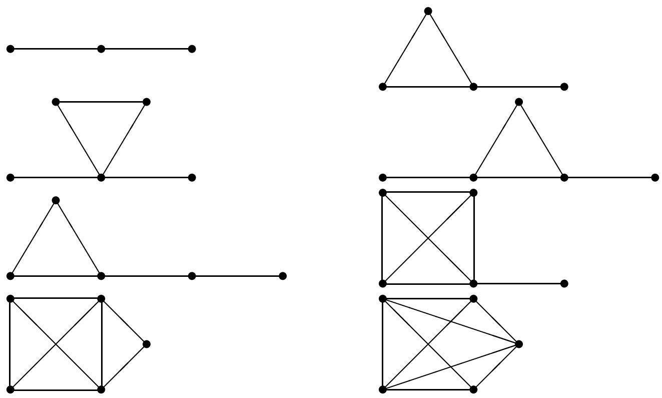
$$|S| \geq \sum_{k=1}^{n} |U_k| + \frac{n(n-3)}{2} - \sum_{j=2}^{n} \sum_{k=1}^{j-1} d(|U_j|, |U_k|).$$

This is easy to prove by induction using the fact $|S| \geq \left|\bigcup_{k=1}^{n} U_k\right| - 1$.

The proposition is also true with $d(|U_j|, |U_k|)$ replaced by $|U_j \cap U_k|$. This quantity, however, is usually not known for a given graph. The corollary gives only the orders of U_k for $k = 1, \ldots, n$, where $n = \sum_{r=1}^{m} v_r$.

Also note that the proposition is already true for $n = 1$ interpreting the second sum as 0.

Together with the last corollary the proposition helps to exclude certain graphs which cannot be associated with generating sets of groups, such as the following.



Graphs not fulfilling the subgroup condition

Take for example the first graph in the second column. In this case $c_1 = 4$, $c_2 = 4$, $c_3 = 1$, and $c_4 = 0$. This yields $v_1 = 0$, $v_2 = 1$, $v_3 = 1$, $v_4 = 0$, and $n = 2$ but $|S| = 4 < 5 = (3 + 4) - 1 - d(3, 4)$ which contradicts the proposition. The other examples were obtained by similar reasoning.

3. Proof of Theorems 1.1 and 1.2

Let S be a symmetric generating set of a finitely-generated group Γ. To prove Theorem 1.1 the mapping $\Phi : s \mapsto s^{-1}$ of S to itself is employed. Note that $N(s^{-1}) = N(s)$ because $(s, t) \in K$ if and only if $(s^{-1}, s^{-1}t) \in K$. So the mapping $t \mapsto s^{-1}t$ from the neighbours of s to the neighbours of s^{-1} is one-to-one. For elements of order 2 even the following holds.

Lemma 3.1: *If $s \in S$ has order 2, then the number of neighbours of s is even.*

Proof: As s has order 2, for the edges $(s, t) \in K$ if and only if $(s, st) \in K$. So the mapping $t \mapsto st$ maps neighbours of s to themselves. Also $st \neq t$ for every neighbour t; otherwise $s = 1 \notin S$. So the neighbours of s come in disjoint pairs $\{t, st\}$, and this shows that $N(s) \equiv 0 \bmod 2$. □

Now Theorem 1.1 can be proven.

Proof: Let $n \equiv 1 \bmod 2$, then the mapping Φ maps $N^{-1}(\{n\})$ to itself. By the last lemma $N^{-1}(\{n\})$ contains no element of order 2. So there is a disjoint partition of $N^{-1}(\{n\})$ into pairs $\{s, s^{-1}\}$ and again this shows that $|N^{-1}(\{n\})| \equiv 0 \bmod 2$. □

Now Theorem 1.2 will be proven.

Proof: Let M be the set of all sets of r nodes which form complete subgraphs and $M_1 = \{C \cup \{1\} : C \in M\}$,

$$\begin{aligned} A &= \{(s_1, \ldots, s_r) : \{s_1, \ldots, s_r\} \in M\}, \\ A_1 &= \{(s_1, \ldots, s_{r+1}) : \{s_1, \ldots, s_{r+1}\} \in M_1\}. \end{aligned}$$

Then the symmetric group S_{r+1} acts on A_1 by

$$\sigma(s_1, \ldots, s_{r+1}) = \left(s_{\sigma(1)}, \ldots, s_{\sigma(r+1)}\right).$$

Let the mapping $P : A_1 \to A$ be defined by

$$P(s_1, \ldots, s_{r+1}) = \left(s_{r+1}^{-1}s_1, \ldots, s_{r+1}^{-1}s_r\right).$$

Note that P is surjective. Also via the mapping P, the symmetric group S_{r+1} acts on A by

$$\sigma(P(s_1, \ldots, s_{r+1})) = P(\sigma(s_1, \ldots, s_{r+1}))$$

for $\sigma \in S_{r+1}$. This is well-defined. Indeed, if $P(s_1, \ldots, s_{r+1}) = P(t_1, \ldots, t_{r+1})$, then

$$\begin{aligned} & P(\sigma(s_1, \ldots, s_r, s_{r+1})) \\ = & \left(s_{\sigma(r+1)}^{-1}s_{\sigma(1)}, \ldots, s_{\sigma(r+1)}^{-1}s_{\sigma(r)}\right) \\ = & \left(\left(s_{r+1}^{-1}s_{\sigma(r+1)}\right)^{-1} s_{r+1}^{-1}s_{\sigma(1)}, \ldots, \left(s_{r+1}^{-1}s_{\sigma(r+1)}\right)^{-1} s_{r+1}^{-1}s_{\sigma(r)}\right) \\ = & \left(\left(t_{r+1}^{-1}t_{\sigma(r+1)}\right)^{-1} t_{r+1}^{-1}t_{\sigma(1)}, \ldots, \left(t_{r+1}^{-1}t_{\sigma(r+1)}\right)^{-1} t_{r+1}^{-1}t_{\sigma(r)}\right) \\ = & P(\sigma(t_1, \ldots, t_r, t_{r+1})). \end{aligned}$$

Considering S_r as a subgroup of S_{r+1} fixing the point $r+1$, we see

$$P(\sigma(s_1, \ldots, s_{r+1})) = \left(s_{r+1}^{-1}s_{\sigma(1)}, \ldots, s_{r+1}^{-1}s_{\sigma(r)}\right) \neq P(s_1, \ldots, s_{r+1})$$

for $\sigma \in S_r \setminus \{1\}$. Let S_r act on the orbit $S_{r+1}V$ for $V \in A$. The above shows that every orbit of the S_r-action has length $r!$ and hence $r!$ is a divisor of $|S_{r+1}V|$.

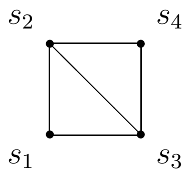
Let $B \subset A$ be a set of representatives such that $A = \bigcup_{V \in B} S_{r+1}V$ is a disjoint decomposition into orbits. Then $r!c_r = |A| = \sum_{V \in B} |S_{r+1}V|$. If $|S_{r+1}V| \neq r!$ then since $|S_{r+1}V|$ divides $(r+1)!$, there exists a prime p_j dividing $r+1$ such that $r!p_j$ divides $|S_{r+1}V|$. If $V = (s_1, \ldots, s_r)$ and $|S_{r+1}V| = r!$, then the set $U = \{1, s_1, \ldots, s_r\}$ is a subgroup of order $r+1$ as $s^{-1}t \in U$ for all $s, t \in U$. Hence $c_r = u_r + \sum_{j=1}^{m} p_j k_j$ for some $k_j \geq 0$. $\square$

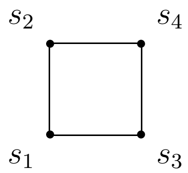
A. Appendix

With the help of the two conditions given by Theorems 1.1 and 1.2 the following complete list of graphs with ≤ 5 nodes together with the associated groups and generating sets was determined. For the graphs not excluded by the preceding theorems a combinatorial case-by-case analysis as below for the remaining graphs excluded some of the rest of the graphs with ≤ 5 nodes. For the others this yielded groups whose generating set is associated with the given graph.

The groups are given in terms of generators and relations. The group $\Gamma = \langle s_1, \ldots, s_n : r_1, \ldots, r_m \rangle$ is the factor group of the free group with generators $s_1, \ldots, s_n$ by the normal subgroup generated by the elements $r_1, \ldots, r_m$ of the free group. Note that each given group Γ is universal in the sense that it is the largest possible group of its type associated with the given graph, and every other group of the same type corresponding to the given graph is a factor group of Γ.

Obviously every complete graph is contained in the list; indeed, for any finite group Γ we may take $S = \Gamma \setminus \{1\}$. On 4 nodes, the graphs not excluded by the conditions above are the following:



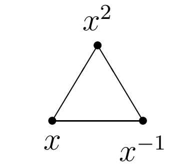
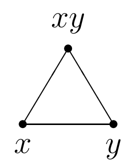


In the first case $s_3 = s_2^{-1}$ by Lemma 3.1. As $|K| = 10$ there are $n = 4$ elements of order 3 by Corollary 1.3 and so $s_4 = s_1^{-1}$. But then $(s_1, s_4) \in K$ since $s_1^{-1}s_4 = s_4^2 = s_1 \in S$, a contradiction.

In the second case $|K| = 8$ so by Corollary 1.3 there are two elements of order 3, say s and s^{-1}, and these must be adjacent as $s^{-1}s^{-1} = s \in S$. Without loss of generality $s_2 = s$ and $s_4 = s^{-1}$, and as $(s_4, s_3) \in K$, we have $ss_3 = s_4^{-1}s_3 \in S$. The only possibility is $ss_3 = s_1$. If s_1 has order 2 then so does s_3 and we have the dihedral group D_3 of order 6, and otherwise $s_3 = s_1^{-1}$ and then $s = s_1 s_3^{-1} = s_1^2$ so $\Gamma = \langle S \rangle = \langle s_1 \rangle$ must be cyclic of order 6.

In the third case $s_4 = s_1^{-1}$. Since $|K| = 6$ there are no elements of order 3. Also either $s_3^{-1} = s_2$ and $s_3^{-1}s_2 = s_1$, or $s_3^{-1} = s_3$ and $s_3^{-1}s_2 = s_4$. The other possibilities lead to contradictions. The first possibility yields $s_1 = s_2^2$ and so $\Gamma = \langle S \rangle = \langle s_2 \rangle \cong \mathbf{Z}$. The second possibility yields $\Gamma = \langle S \rangle = \langle s_2, s_3 \rangle \cong D_\infty$.

Proofs of this classification for the remaining graphs with 5 nodes and less than 6 edges by similar (but more involved) reasoning can be found in [5].

<table>
<tr><th>Group</th><th>Graph</th></tr>
<tr><td>$\Gamma = \langle x : x^3 \rangle \cong \mathbf{Z}/3\mathbf{Z}$
$S = \{x^{\pm 1}\}$</td><td>
</td></tr>
<tr><td>$\Gamma = \langle x : x^4 \rangle \cong \mathbf{Z}/4\mathbf{Z}$
$S = \{x^{\pm 1}, x^2\}$</td><td>
</td></tr>
<tr><td>$\Gamma = \left\langle x, y : x^2, y^2, (xy)^2 \right\rangle \cong D_2$
$\cong \langle x : x^2 \rangle \times \langle y : y^2 \rangle$
$\cong (\mathbf{Z}/2\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z})$
$S = \{x, y, xy\}$</td><td>
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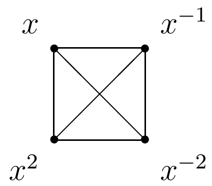
<tr><td>$\Gamma = \langle x : x^5 \rangle \cong \mathbf{Z}/5\mathbf{Z}$
$S = \{x^{\pm 1}, x^{\pm 2}\}$</td><td>
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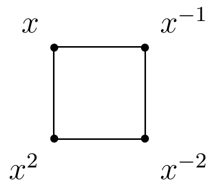
<tr><td>$\Gamma = \langle x : x^6 \rangle \cong \mathbf{Z}/6\mathbf{Z}$
$S = \{x^{\pm 1}, x^{\pm 2}\}$</td><td>
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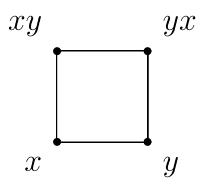
<tr><td>$\Gamma = \left\langle x, y : x^2, y^2, (xy)^3 \right\rangle \cong D_3$
$S = \{x, y, xy, yx\}$</td><td>
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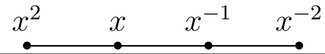
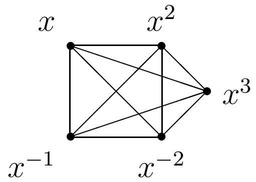
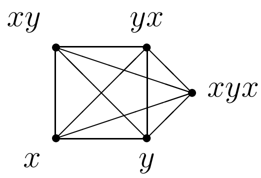
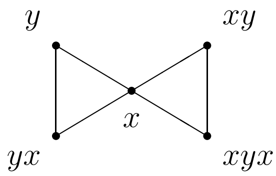
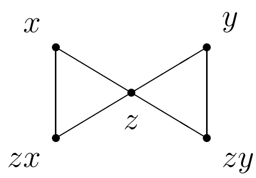
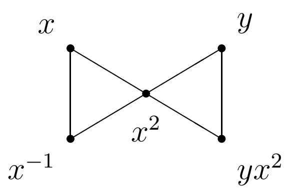
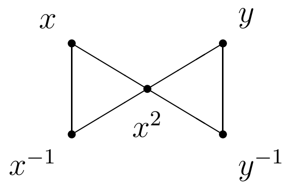
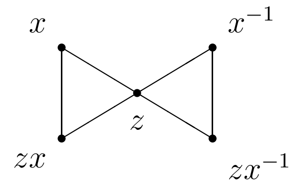
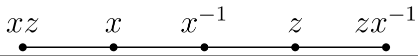
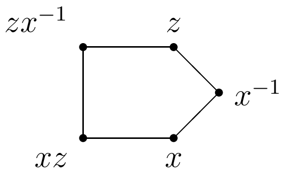
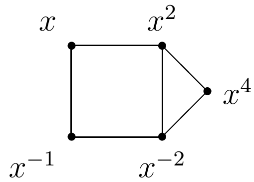
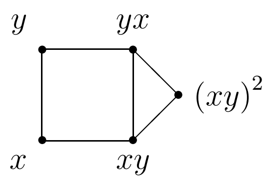
Group	Graph
$\Gamma = \langle x \rangle \cong \mathbf{Z}$ $S = \{x^{\pm 1}, x^{\pm 2}\}$	
$\Gamma = \langle x, y : x^2, y^2 \rangle \cong D_\infty$, $S = \{x, y, xy, yx\}$	
$\Gamma = \langle x : x^6 \rangle \cong \mathbf{Z}/6\mathbf{Z}$ $S = \{x^{\pm 1}, x^{\pm 2}, x^3\}$	
$\Gamma = \left\langle x, y : x^2, y^2, (xy)^3 \right\rangle \cong D_3$ $S = \{x, y, xy, yx, xyx\}$	
$\Gamma = \langle x, y : x^2, y^2 \rangle \cong D_\infty$ $S = \{x, y, xy, yx, xyx\}$	
$\Gamma = \left\langle x, y, z : x^2, y^2, z^2, (zx)^2, (zy)^2 \right\rangle$ $\cong \langle x, y : x^2, y^2 \rangle \times \langle z : z^2 \rangle$ $\cong D_\infty \times (\mathbf{Z}/2\mathbf{Z})$ $S = \{x, y, z, zx, zy\}$	
$\Gamma = \left\langle x, y : x^4, y^2, (yx^2)^2 \right\rangle$ $S = \{x^{\pm 1}, x^2, y, yx^2\}$	

<table>
<tr><th>Group</th><th>Graph</th></tr>
<tr><td>$\Gamma = \langle x, y : x^4, y^4, x^2y^2 \rangle$
$S = \{x^{\pm 1}, y^{\pm 1}, x^2\}$</td><td></td></tr>
<tr><td>$\Gamma = \langle x, z : z^2, zxzx^{-1} \rangle$
$\cong \langle x \rangle \times \langle z : z^2 \rangle$
$\cong \mathbf{Z} \times (\mathbf{Z}/2\mathbf{Z})$
$S = \{x^{\pm 1}, z, zx, zx^{-1}\}$</td><td></td></tr>
<tr><td>$\Gamma = \langle x, z : x^3, z^2 \rangle \cong \mathrm{PSL}(2, \mathbf{Z})$
$S = \{x^{\pm 1}, z, xz, zx^{-1}\}$</td><td></td></tr>
<tr><td>$\Gamma = \left\langle x, z : x^3, z^2, (xz)^3 \right\rangle$
$\cong \mathrm{PSL}(2, \mathbf{Z}/3\mathbf{Z})$
$S = \{x^{\pm 1}, z, xz, zx^{-1}\}$</td><td></td></tr>
<tr><td>$\Gamma = \langle x : x^8 \rangle \cong \mathbf{Z}/8\mathbf{Z}$
$S = \{x^{\pm 1}, x^{\pm 2}, x^4\}$</td><td></td></tr>
<tr><td>$\Gamma = \left\langle x, y : x^2, y^2, (xy)^4 \right\rangle \cong D_4$
$S = \left\{x, y, xy, yx, (xy)^2\right\}$</td><td></td></tr>
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