# ON COMBINATORIAL IDENTITIES OF ENGBERS AND STOCKER

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**Abstract.** We extend two combinatorial identitites published by Engbers and Stocker in 2016. Among others, we prove that if $b$, $n$ and $r$ are integers such that $b \geq 1$ and $n-1 \geq r \geq 0$, then

$$\sum_{k=0}^{r} \binom{r}{k}^2 \binom{k+n}{2r+b} = \sum_{k=0}^{n-1} \binom{k}{r}^2 \binom{n-k}{b-1}.$$

The special case $b = 1$ is due to Engbers and Stocker.

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## 1. Introduction and statement of results

The work on this note has been inspired by an interesting research paper published by Engbers and Stocker [1] in 2016. The authors use combinatorial techniques to show that the identities

$$\sum_{k=0}^{r} \binom{r}{k}^2 \binom{k+n}{2r+1} = \sum_{k=r}^{n-1} \binom{k}{r}^2 \tag{1}$$

and

$$\sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \binom{n}{k+1} = \sum_{k=r}^{n-1} \binom{k}{r}^2 \tag{2}$$

are valid for all integers $n$ and $r$ with $n-1 \geq r \geq 0$. Actually, they prove a bit more. They present summation formulas involving $\sum_{k=r}^{n-1} \binom{k}{r}^s$, where $s$ is a natural number. The identities (1) and (2) turn out to be the most attractive special cases.

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Here, we provide a different kind of extension. We study the sums

$$S_{r,n}(b)=\sum_{k=0}^{r}\binom{r}{k}^2\binom{k+n}{2r+b}$$

and

$$T_{r,n}(b)=\sum_{k=r}^{2r}\binom{2(k-r)}{k-r}\binom{k}{2r-k}\binom{n}{k+b},$$

where $b$ is an integer. In the next sections we use the concept of generating functions to prove new extensions of (1) and (2). Our extension of (1) reads as follows.

**Theorem 1.** *Let $b$, $n$ and $r$ be integers with $n-1\ge r\ge 0$.*
(i) *If $b\ge 1$, then*

$$S_{r,n}(b)=\sum_{k=r}^{n-1}\binom{k}{r}^2\binom{n-k-1}{b-1}.$$

(ii) *If $b\le 0$, then*

$$S_{r,n}(b)=\sum_{k=0}^{-b}(-1)^{-b-k}\binom{k+n}{r}^2\binom{-b}{k}.$$

The case $b=1$ gives (1) whereas the special cases $b=0$ and $b=-1$ lead to the elegant identities

$$\sum_{k=0}^{r}\binom{r}{k}^2\binom{k+n}{2r}=\binom{n}{r}^2 \tag{3}$$

and

$$\sum_{k=0}^{r}\binom{r}{k}^2\binom{k+n}{2r-1}=\binom{n+1}{r}^2-\binom{n}{r}^2. \tag{4}$$

The sum

$$U_{r,n}(b)=\sum_{k=1}^{r}\binom{r}{k-1}\binom{r}{k}\binom{k+n}{2r+b}$$

is closely related to $S_{r,n}(b)$. We apply

$$2\binom{r}{k-1}\binom{r}{k}=\binom{r+1}{k}^2-\binom{r}{k-1}^2-\binom{r}{k}^2$$

and obtain the representation

$$U_{r,n}(b)=\frac{1}{2}\Big(S_{r+1,n}(b-2)-S_{r,n+1}(b)-S_{r,n}(b)\Big). \tag{5}$$

Using (5) with $b = 1, 0, -1$, respectively, we conclude from Theorem 1 that the following counterparts of (1), (3) and (4) are valid:

$$\sum_{k=1}^{r} \binom{r}{k-1}\binom{r}{k}\binom{k+n}{2r+1} = \binom{n}{r}\binom{n}{r+1} - \sum_{k=r}^{n-1} \binom{k}{r}^2,$$

$$\sum_{k=1}^{r} \binom{r}{k-1}\binom{r}{k}\binom{k+n}{2r} = \binom{n}{r-1}\binom{n+1}{r+1},$$

$$\sum_{k=1}^{r} \binom{r}{k-1}\binom{r}{k}\binom{k+n}{2r-1} = \binom{n+1}{r-1}\binom{n+2}{r+1} - \binom{n}{r-1}\binom{n+1}{r+1}.$$

In Section 3, we prove the following generalization of (2).

**Theorem 2.** *Let $b$, $n$ and $r$ be integers with $n-1 \geq r \geq 0$.*
(i) *If $b \geq 1$, then*

$$T_{r,n}(b) = \sum_{k=r}^{n-1} \binom{k}{r}^2 \binom{n-k-1}{b-1}.$$

(ii) *If $b \leq 0$, then*

$$T_{r,n}(b) = \sum_{k=0}^{-b} (-1)^{-b-k} \binom{k+n}{r}^2 \binom{-b}{k}.$$

In particular, the special cases $b = 1$ and $b = 0, -1$ lead to (2) and

$$\sum_{k=r}^{2r} \binom{2(k-r)}{k-r}\binom{k}{2r-k}\binom{n}{k} = \binom{n}{r}^2,$$

$$\sum_{k=r}^{2r} \binom{2(k-r)}{k-r}\binom{k}{2r-k}\binom{n}{k-1} = \binom{n+1}{r}^2 - \binom{n}{r}^2,$$

respectively.

## 2. Proof of Theorem 1

We define

$$F_b(x,u) = \sum_{n,r \geq 0} S_{r,n}(b) x^n u^r$$

and

$$q_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k.$$

Then, see [3, pp. 78, 81],

$$\sum_{n\ge0} u^n q_n(x) = \frac{1}{\sqrt{1-2(1+x)u+(1-x)^2u^2}}.$$

We obtain

$$\sum_{n\ge0} x^n \sum_{k=0}^{r} \binom{r}{k}^2 \binom{k+n}{2r} = \sum_{k=0}^{r} \binom{r}{k}^2 \frac{x^{2r-k}}{(1-x)^{2r+1}} = \frac{x^r}{(1-x)^{2r+1}} q_r(x)$$

and furthermore

$$\begin{aligned}
F_0(x,u) &= \sum_{r\ge0} u^r \sum_{n\ge0} x^n \sum_{k=0}^{r} \binom{r}{k}^2 \binom{k+n}{2r} = \frac{1}{1-x} \sum_{r\ge0} \left( \frac{ux}{(1-x)^2} \right)^r q_r(x) \\
&= \frac{1}{1-x} \frac{1}{\sqrt{1-2(1+x)\frac{ux}{(1-x)^2}+(1-x)^2\frac{u^2x^2}{(1-x)^4}}} \\
&= \frac{1}{\sqrt{1-2(1+u)x+(1-u)^2x^2}} \\
&= \sum_{n,r\ge0} \binom{n}{r}^2 x^n u^r.
\end{aligned}$$

Comparing the coefficients of $x^n u^r$ we find the identity

$$S_{r,n}(0) = \sum_{k=0}^{r} \binom{r}{k}^2 \binom{k+n}{2r} = \binom{n}{r}^2.$$

Now, let $b \ge 1$. Applying the following variant of the Vandermonde formula, see [2, p. 169],

$$\binom{k+n}{2r+b} = \sum_{j=0}^{n-1} \binom{k+j}{2r} \binom{n-j-1}{b-1} \quad (0 \le k \le r)$$

we obtain

$$\begin{aligned}
S_{r,n}(b) &= \sum_{k=0}^{r} \binom{r}{k}^2 \sum_{j=0}^{n-1} \binom{k+j}{2r} \binom{n-j-1}{b-1} \\
&= \sum_{j=0}^{n-1} \binom{n-j-1}{b-1} S_{r,j}(0) = \sum_{j=0}^{n-1} \binom{n-j-1}{b-1} \binom{j}{r}^2.
\end{aligned}$$

Next, let $b \le 0$. Using the Vandermonde type identity, see [2, p. 169],

$$\binom{k+n}{2r+b} = \sum_{j=0}^{-b} \binom{-b}{j}\binom{k+j+n}{2r}(-1)^{-b-j}$$

we get

$$\begin{aligned} S_{r,n}(b) &= \sum_{k=0}^{r} \binom{r}{k}^2 \sum_{j=0}^{-b} \binom{-b}{j}\binom{k+j+n}{2r}(-1)^{-b-j} \\ &= \sum_{j=0}^{-b} \binom{-b}{j}(-1)^{-b-j} S_{r,j+n}(0) = \sum_{j=0}^{-b} \binom{-b}{j}(-1)^{-b-j}\binom{j+n}{r}^2. \end{aligned}$$

This completes the proof of Theorem 1.

## 3. Proof of Theorem 2

As before, we consider the bivariate generating function

$$G_b(x,u) = \sum_{n,r\ge 0} T_{r,n}(b) x^n u^r.$$

We have, see [3, page 73]:

$$\sum_{n\ge 0} u^n \sum_{0\le k\le n/2} \binom{2k}{k}\binom{n}{2k} x^{2k}(1-2x)^{n-2k} = \frac{1}{\sqrt{[1-(1-2x)u]^2 - 4x^2u^2}}.$$

Next, we replace $x$ by $\sqrt{x}/(1+2\sqrt{x})$ and $u$ by $(1+2\sqrt{x})u$. This leads to

$$\sum_{n\ge 0} u^n \sum_{0\le k\le n/2} \binom{2k}{k}\binom{n}{2k} x^k = \frac{1}{\sqrt{(1-u)^2 - 4xu^2}}.$$

We set $t = z/(1-z)$ and apply

$$\sum_{k=r}^{2r} \binom{2(k-r)}{k-r}\binom{k}{2r-k} \sum_{n\ge 0}\binom{n}{k} z^n = \sum_{k=r}^{2r} \binom{2(k-r)}{k-r}\binom{k}{2r-k} \frac{z^k}{(1-z)^{k+1}}.$$

Then we obtain

$$\begin{aligned} G_0(z,u) &= \sum_{r\ge 0} u^r \sum_{k=r}^{2r} \binom{2(k-r)}{k-r}\binom{k}{2r-k}\frac{z^k}{(1-z)^{k+1}} \\ &= \frac{1}{1-z}\sum_{r\ge 0} u^r \sum_{k=r}^{2r} \binom{2(k-r)}{k-r}\binom{k}{2r-k} t^k \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1-z}\sum_{k\ge 0}\sum_{r\ge 0} u^{k-r}\binom{2r}{r}\binom{k}{2r}t^k \\
&= \frac{1}{1-z}\sum_{k\ge 0}(ut)^k\sum_{r\ge 0} u^{-r}\binom{2r}{r}\binom{k}{2r} \\
&= \frac{1}{1-z}\frac{1}{\sqrt{(1-ut)^2-4ut^2}} \\
&= \frac{1}{\sqrt{1-2(1+u)z+(1-u)^2z^2}} \\
&= \sum_{n,r\ge 0}\binom{n}{r}^2 z^n u^r.
\end{aligned}
$$

We compare the coefficients of $z^n u^r$ and find

$$T_{r,n}(0) = \sum_{k=r}^{2r}\binom{2(k-r)}{k-r}\binom{k}{2r-k}\binom{n}{k} = \binom{n}{r}^2.$$

Now, let $b \ge 1$. Using

$$\binom{n}{k+b} = \sum_{j=0}^{n-1}\binom{j}{k}\binom{n-j-1}{b-1}$$

leads to

$$
\begin{aligned}
T_{r,n}(b) &= \sum_{k=r}^{2r}\binom{2(k-r)}{k-r}\binom{k}{2r-k}\sum_{j=0}^{n-1}\binom{j}{k}\binom{n-j-1}{b-1} \\
&= \sum_{j=0}^{n-1}\binom{n-j-1}{b-1}T_{r,j}(0) = \sum_{j=0}^{n-1}\binom{n-j-1}{b-1}\binom{j}{r}^2.
\end{aligned}
$$

Next, let $b \le 0$. Since

$$\binom{n}{k+b} = \sum_{j=0}^{-b}\binom{-b}{j}\binom{j+n}{k}(-1)^{-b-j},$$

we obtain

$$
\begin{aligned}
T_{r,n}(b) &= \sum_{k=r}^{2r}\binom{2(k-r)}{k-r}\binom{k}{2r-k}\sum_{j=0}^{-b}\binom{-b}{j}\binom{j+n}{k}(-1)^{-b-j} \\
&= \sum_{j=0}^{-b}\binom{-b}{j}(-1)^{-b-j}T_{r,j+n}(0) = \sum_{j=0}^{-b}\binom{-b}{j}(-1)^{-b-j}\binom{j+n}{r}^2.
\end{aligned}
$$

The proof of Theorem 2 is complete.

## REFERENCES


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