A COMBINATORIAL STUDY OF TWO-PERIODIC RANDOM WALKS

WOLFGANG PANNY AND HELMUT PRODINGER‡

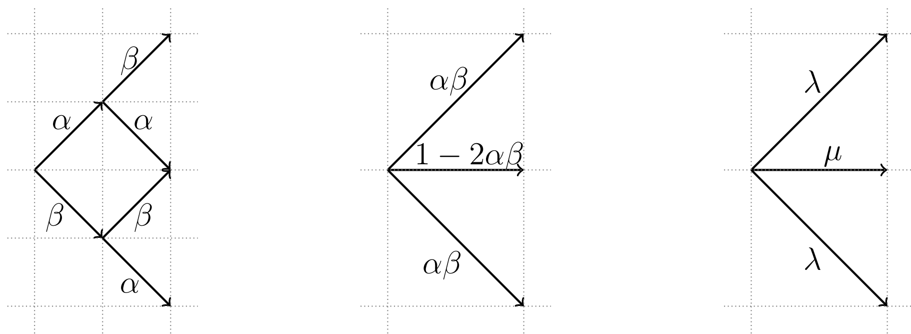
Abstract. Two-periodic random walks have up-steps and down-steps of one unit as usual, but the probability of an up-step is α after an even number of steps and $\beta = 1 - \alpha$ after an odd number of steps, and reversed for down-steps. This concept was studied by Böhm and Hornik [1]. We complement this analysis by using methods from (analytic) combinatorics. By using two steps at once, we can reduce the analysis to the study of Motzkin paths, with up-steps, down-steps, and level-steps. Using a proper substitution, we get the generating functions of interest in an explicit and neat form. The parameters that are discussed here are the (one-sided) maximum (already studied by Böhm and Hornik [1]) and the two-sided maximum. For the asymptotic evaluation of the average value of the two-sided maximum after n random steps, more sophisticated methods from complex analysis (Mellin transform, singularity analysis) are required. The approach to transfer the analysis to Motzkin paths is of course not restricted to the two parameters under consideration.

1. Introduction

Böhm and Hornik [1] have studied two-periodic random walks, with up-steps and down-steps of one unit as usual, but the probability of an up-step is α after an even number of steps and $\beta = 1 - \alpha$ after an odd number of steps, and reversed for down-steps. See [1] for motivation and references. In the present paper we want to use a different method to prove new results and rederive some others. So, this paper can be seen as a companion paper to [1]. As references for the approach we choose we cite [6, 7, 5]. In [1], a fair amount of space was dedicated to the study of the maximum of such a random walk. The model under consideration is that the random walk (lattice path) starts in the origin and may end anywhere after n steps. We plan, however, to move to different models (e. g., non-negative paths) in future publications. The following figure will be useful to understand our procedure.

We look at pairs of steps, and, since after an even number of steps, the random walk can only end at an even level, we might as well reduce the grid to points with both coordinates being even. In this way, we get a modified random walk which resembles Motzkin paths (up-step, down-step, level-step), with suitable weights. We use the abbreviations $\lambda = \alpha\beta$ and $\mu = 1 - 2\lambda$. Studying the modified walk will lead to enumeration results for the original walk, after $2n$ steps. It is, however, only a small modification, to deal with an odd number of steps, since we just add one single step at the end, and this can be described. The parameter "maximum" is translated to the maximum of the modified path, but with a small twist. We will compute (probability)

‡This author was supported by an incentive grant of the National Research Foundation of South Africa.



generating functions for walks with maximum $\le h$. If h is an odd number, $h = 2H+1$, then this is directly given by the maximum of the modified path being $\le H$, but for even $h = 2H$, we must be careful that the original path, during odd numbered steps, does not reach the level $h+1$. So, for the modified path, on level H, we can make a level step, but not with the full probability μ, but only with β^2, which is the weight of a down-up double step in the original path. The same principle applies, mutatis mutandis, when we bound the random walks from below.

We will find generating functions for walks bounded from above and below, with modifications when necessary. For each level that the walk can reach, there will be one generating function, and they satisfy a system of linear equations that can be solved explicitly using Cramer's rule. The determinants that show up satisfy a second order recursion, whence there is an explicit expression for them. One essential step within our method is the substitution

$$z = \frac{v}{\lambda + \mu v + \lambda v^2}.$$

Using this, all functions become rational, and manipulations are much easier. When we are not interested in the level where the walks end, we simply sum. Sending the lower bound to $-\infty$ leads eventually to walks that are only bounded from above. From this the expectation of the maximum parameter can be explicitly derived, and more results as well if desired.

We will also deal with a new parameter , which is very natural here. If the (original) walk is $(0, s_0 = 0), (1, s_1), \dots, (n, s_n)$, then we consider $\max_{0\le i\le n}\{|s_i|\}$ instead of $\max_{0\le i\le n}\{s_i\}$ (maximum). We will call this the two-sided maximum. For this, we compute generating functions of walks that "live" in a strip $-h \le \cdots \le h$. Once again, for odd h, this is easy, but for even h we need the modification at the top and bottom levels. This parameter is more complicated than the maximum itself since it requires more advanced techniques (Mellin transform, singularity analysis) for the asymptotic evaluation.

2. The maximum

We start with paths (in the original sense), "living" in the strip $-(2k+1) \le \cdots \le 2h+1$. For the modified paths, this means $-k \le \cdots \le h$. Eventually, we will send k to infinity. According to the discussion in the Introduction, this leads to a system of linear equations for

$$\varphi_i(z) = \sum_{n\ge 0} [\text{probability that an admissible path ends at level } i \text{ after } n \text{ steps}] z^n :$$

$$\begin{bmatrix} \varphi_h \\ \varphi_{h-1} \\ \vdots \\ \varphi_0 \\ \vdots \\ \varphi_{-(k-1)} \\ \varphi_{-k} \end{bmatrix} = z \begin{bmatrix} \mu & \lambda & & & & & \\ \lambda & \mu & \lambda & & & & \\ & \lambda & \mu & \lambda & & & \\ & & & \ddots & & & \\ & & & \lambda & \mu & \lambda & \\ & & & & \lambda & \mu & \lambda \\ & & & & & \lambda & \mu \end{bmatrix} \begin{bmatrix} \varphi_h \\ \varphi_{h-1} \\ \vdots \\ \varphi_0 \\ \vdots \\ \varphi_{-(k-1)} \\ \varphi_{-k} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

These functions also depend on h and k, but we avoid a clumsy notation for this. A rearrangement leads to

$$\begin{bmatrix} 1-\mu z & -\lambda z & & & & & \\ -\lambda z & 1-\mu z & -\lambda z & & & & \\ & -\lambda z & 1-\mu z & -\lambda z & & & \\ & & & \ddots & & & \\ & & & -\lambda z & 1-\mu z & -\lambda z & \\ & & & & -\lambda z & 1-\mu z & -\lambda z \\ & & & & & -\lambda z & 1-\mu z \end{bmatrix} \begin{bmatrix} \varphi_h \\ \varphi_{h-1} \\ \vdots \\ \varphi_0 \\ \vdots \\ \varphi_{-(k-1)} \\ \varphi_{-k} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

This system will be solved using Cramer's rule. For that, we need the following determinants:

$$a_i = \begin{vmatrix} 1-\mu z & -\lambda z & & & \\ -\lambda z & 1-\mu z & -\lambda z & & \\ & & \ddots & & \\ & & -\lambda z & 1-\mu z & -\lambda z \\ & & & -\lambda z & 1-\mu z \end{vmatrix}$$

with i rows. Expanding the determinant with respect to the first row, say, leads to

$$a_i = (1-\mu z)a_{i-1} - \lambda^2 z^2 a_{i-2}$$

with $a_0 = 1$, $a_1 = 1-\mu z$. Using standard methods to solve the recursion, together with the substitution

$$z = \frac{v}{\lambda + \mu v + \lambda v^2} \quad \leftrightarrow \quad v = \frac{1-\mu z - \sqrt{(1-z)(1-z(1-4\lambda))}}{2\lambda z}$$

leads to the explicit (and appealing!) answer

$$a_n = \frac{1-v^{2n+2}}{1-v^2}\left(\frac{\lambda}{\lambda+\mu v+\lambda v^2}\right)^n.$$

Note that the root with the negative sign has the property $z \to 0 \Leftrightarrow v \to 0$.

With these quantities, the desired generating functions can be computed:

$$\varphi_i = \frac{(\lambda z)^i a_{h-i} a_k}{a_{h+k+1}}, \quad i = 0, \dots, h,$$

$$\varphi_{-i} = \frac{(\lambda z)^i a_h a_{k-i}}{a_{h+k+1}}, \quad i = 0, \dots, k.$$

This leads to

$$\varphi_i = \frac{\lambda + \mu v + \lambda v^2}{\lambda} \frac{(1 - v^{2k+2})}{(1 - v^2)(1 - v^{2(h+k+1)+2})} (v^i - v^{2h-i+2})$$

and

$$\varphi_{-i} = \frac{\lambda + \mu v + \lambda v^2}{\lambda} \frac{(1 - v^{2h+2})}{(1 - v^2)(1 - v^{2(h+k+1)+2})} (v^i - v^{2k-i+2}).$$

Further,

$$\sum_{i=0}^{h} \varphi_i = \frac{\lambda + \mu v + \lambda v^2}{\lambda} \frac{(1 - v^{2k+2})(1 - v^{h+1})(1 - v^{h+2})}{(1 - v)(1 - v^2)(1 - v^{2(h+k+1)+2})}$$

and

$$\sum_{i=1}^{k} \varphi_{-i} = \frac{\lambda + \mu v + \lambda v^2}{\lambda} \frac{v(1 - v^{2h+2})(1 - v^k)(1 - v^{k+1})}{(1 - v)(1 - v^2)(1 - v^{2(h+k+1)+2})}.$$

The sum leads to the generating function of all walks that live in the strip:

$$\sum_{i=0}^{h} \varphi_i + \sum_{i=1}^{k} \varphi_{-i} = \frac{\lambda + \mu v + \lambda v^2}{\lambda} \frac{1 - v^{h+1} - v^{k+1} + v^{2h+k+3} + v^{2k+h+3} - v^{2h+2k+4}}{(1 - v)^2 (1 - v^{2(h+k+1)+2})}. \quad (1)$$

Now we let $k \to \infty$ and get the generating function of paths with maximum $\le 2h + 1$ (original paths, even number of steps).

$$M^{[\le 2h+1]}(z) = \frac{(\lambda + \mu v + \lambda v^2)(1 - v^{h+1})}{\lambda (1 - v)^2}.$$

The limit for $h \to \infty$ leads to all paths:

$$\frac{1}{1 - z} = \frac{(\lambda + \mu v + \lambda v^2)}{\lambda (1 - v)^2}.$$

Taking differences, we find the generating function of paths with maximum $> 2h + 1$:

$$M^{[> 2h+1]}(z) = \frac{(\lambda + \mu v + \lambda v^2) v^{h+1}}{\lambda (1 - v)^2}.$$

Now we move to the more tricky case that the maximum is $\le 2h$. It basically means $\le h$ for the modified path, but with a twist. The probability for a level step at the top level is β^2, not the usual μ. The system to be solved is

$$\begin{bmatrix} 1-\beta^2 z & -\lambda z & & & & & \\ -\lambda z & 1-\mu z & -\lambda z & & & & \\ & -\lambda z & 1-\mu z & -\lambda z & & & \\ & & & \ddots & & & \\ & & & -\lambda z & 1-\mu z & -\lambda z & \\ & & & & -\lambda z & 1-\mu z & -\lambda z \\ & & & & & -\lambda z & 1-\mu z \end{bmatrix} \begin{bmatrix} \varphi_h \\ \varphi_{h-1} \\ \vdots \\ \varphi_0 \\ \vdots \\ \varphi_{-(k-1)} \\ \varphi_{-k} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

For that, we need the determinants

$$b_i = \begin{vmatrix} 1-\beta^2 z & -\lambda z & & & \\ -\lambda z & 1-\mu z & -\lambda z & & \\ & & \ddots & & \\ & & -\lambda z & 1-\mu z & -\lambda z \\ & & & -\lambda z & 1-\mu z \end{vmatrix}$$

with i rows. Expanding with respect to the first row leads to

$$\begin{aligned} b_i &= (1-\beta^2 z)a_{i-1} - (\lambda z)^2 a_{i-2} \\ &= \frac{1}{1-v^2}\Big(\frac{\lambda}{\lambda+\mu v+\lambda v^2}\Big)^i \Big[1+\frac{\alpha}{\beta}v - \frac{\alpha}{\beta}v^{2i+1} - v^{2i+2}\Big]. \end{aligned}$$

Hence, the solution can be written as

$$\begin{aligned} \varphi_i &= \frac{(\lambda z)^i b_{h-i} a_k}{b_{h+k+1}}, \quad i = 0,\dots,h, \\ \varphi_{-i} &= \frac{(\lambda z)^i b_h a_{k-i}}{b_{h+k+1}}, \quad i = 0,\dots,k. \end{aligned}$$

This leads to

$$\varphi_i = v^i \frac{\lambda+\mu v+\lambda v^2}{\lambda} \frac{1-v^{2k+2}}{1-v^2} \frac{\big[1+\frac{\alpha}{\beta}v - \frac{\alpha}{\beta}v^{2h-2i+1} - v^{2h-2i+2}\big]}{\big[1+\frac{\alpha}{\beta}v - \frac{\alpha}{\beta}v^{2h+2k+3} - v^{2h+2k+4}\big]}$$

and

$$\varphi_{-i} = v^i \frac{\lambda+\mu v+\lambda v^2}{\lambda} \frac{1-v^{2k-2i+2}}{1-v^2} \frac{\big[1+\frac{\alpha}{\beta}v - \frac{\alpha}{\beta}v^{2h+1} - v^{2h+2}\big]}{\big[1+\frac{\alpha}{\beta}v - \frac{\alpha}{\beta}v^{2h+2k+3} - v^{2h+2k+4}\big]}.$$

By summing,

$$\begin{aligned} \sum_{i=0}^h \varphi_i &= \frac{\lambda+\mu v+\lambda v^2}{\lambda} \frac{1-v^{2k+2}}{1-v^2} \frac{\sum_{i=0}^h \big[(1+\frac{\alpha}{\beta}v)v^i - (\frac{\alpha}{\beta}+v)v^{2h-i+1}\big]}{\big[1+\frac{\alpha}{\beta}v - \frac{\alpha}{\beta}v^{2h+2k+3} - v^{2h+2k+4}\big]} \\ &= \frac{\lambda+\mu v+\lambda v^2}{\lambda} \frac{(1-v^{2k+2})(1-v^{h+1})}{(1-v)(1-v^2)} \frac{\big[(1+\frac{\alpha}{\beta}v) - (\frac{\alpha}{\beta}+v)v^{h+1}\big]}{\big[1+\frac{\alpha}{\beta}v - \frac{\alpha}{\beta}v^{2h+2k+3} - v^{2h+2k+4}\big]} \end{aligned}$$

and

$$\sum_{i=1}^{k}\varphi_{-i}=\frac{\lambda+\mu v+\lambda v^2}{\lambda}\frac{v(1-v^k)(1-v^{k+1})}{(1-v)(1-v^2)}\frac{\big[(1+\frac{\alpha}{\beta}v)-(\frac{\alpha}{\beta}+v)v^{2h+1}\big]}{\big[1+\frac{\alpha}{\beta}v-\frac{\alpha}{\beta}v^{2h+2k+3}-v^{2h+2k+4}\big]}.$$

To simplify the computations, we let $k\to\infty$ already now and compute

$$\begin{aligned}M^{[\le 2h]}(z)=\sum_{i=0}^{h}\varphi_i+\sum_{i=1}^{\infty}\varphi_{-i}&=\frac{\lambda+\mu v+\lambda v^2}{\lambda}\frac{(1-v^{h+1})}{(1-v)(1-v^2)}\frac{\big[(1+\frac{\alpha}{\beta}v)-(\frac{\alpha}{\beta}+v)v^{h+1}\big]}{(1+\frac{\alpha}{\beta}v)}\\&\quad+\frac{\lambda+\mu v+\lambda v^2}{\lambda}\frac{v}{(1-v)(1-v^2)}\frac{\big[(1+\frac{\alpha}{\beta}v)-(\frac{\alpha}{\beta}+v)v^{2h+1}\big]}{(1+\frac{\alpha}{\beta}v)}\\&=\frac{\lambda+\mu v+\lambda v^2}{\lambda}\frac{\big[(1+\frac{\alpha}{\beta}v)-\frac{1}{\beta}v^{h+1}\big]}{(1+\frac{\alpha}{\beta}v)(1-v)^2}.\end{aligned}$$

The limit $h\to\infty$ is again

$$\frac{\lambda+\mu v+\lambda v^2}{\lambda}\frac{1}{(1+\frac{\alpha}{\beta}v)(1-v)^2}\Big(1+\frac{\alpha}{\beta}v\Big)=\frac{\lambda+\mu v+\lambda v^2}{\lambda}\frac{1}{(1-v)^2},$$

which is a good check. Taking differences, we find

$$M^{[>2h]}(z)=\frac{(\lambda+\mu v+\lambda v^2)v^{h+1}}{\lambda(\beta+\alpha v)(1-v)^2}.$$

Recall that

$$M^{[>2h+1]}(z)=\frac{(\lambda+\mu v+\lambda v^2)v^{h+1}}{\lambda(1-v)^2}.$$

From this we find immediately the generating function of the expectations:

$$\begin{aligned}E(z)&=\sum_{h\ge0}M^{[>h]}(z)=\sum_{h\ge0}M^{[>2h]}(z)+\sum_{h\ge0}M^{[>2h+1]}(z)\\&=\frac{\lambda+\mu v+\lambda v^2}{\lambda(1+v)^2}\left[\frac{1}{\beta+\alpha v}+1\right]\frac{v}{1-v}\\&=\frac{v(\lambda+\mu v+\lambda v^2)(1+\beta+\alpha v)}{\lambda(\beta+\alpha v)(1-v)^3}\\&=\frac{v(\alpha+\beta v)(1+\beta+\alpha v)}{\lambda(1-v)^3}\\&=\frac{2}{\lambda(1-v)^3}-\frac{4-\alpha}{\lambda(1-v)^2}+\frac{2-\alpha^2}{\lambda(1-v)}-1\,.\end{aligned}$$

Before we engage into the study of paths with an odd number of steps, let us get an explicit expression for the expectation. We will use the notion of (weighted) trinomial coefficients [2]

$$\binom{n;\lambda,\mu,\lambda}{k}=[v^k](\lambda+\mu v+\lambda v^2)^n.$$

Then

$$
\begin{aligned}
\mathbb{E}\mathsf{Max}_{2n} &= [z^n]E(z)\\
&= \frac{1}{2\pi i}\oint \frac{dz}{z^{n+1}}\left[\frac{2}{\lambda(1-v)^3}-\frac{4-\alpha}{\lambda(1-v)^2}+\frac{2-\alpha^2}{\lambda(1-v)}-1\right]\\
&= \frac{1}{2\pi i}\oint \frac{(\lambda+\mu v+\lambda v^2)^{n-1}\lambda(1-v^2)}{v^{n+1}}\left[\frac{2}{\lambda(1-v)^3}-\frac{4-\alpha}{\lambda(1-v)^2}+\frac{2-\alpha^2}{\lambda(1-v)}-1\right]dv\\
&= [v^n]\,(\lambda+\mu v+\lambda v^2)^{n-1}\,(1+v)\left[\frac{2}{(1-v)^2}-\frac{4-\alpha}{1-v}+(2-\alpha^2)\right]-\lambda(1-v^2)\\
&= [v^n]\,(\lambda+\mu v+\lambda v^2)^{n-1}\left[\sum_{k\ge1}\left[4kv^k-2(3-\alpha)\right]v^k+(2-\alpha^2)v+\lambda v^2\right]\\
&= \lambda\binom{n-1;\lambda,\mu,\lambda}{n-2}+(\alpha+\lambda)\binom{n-1;\lambda,\mu,\lambda}{n-1}+\sum_{k=1}^{n-1}\binom{n-1;\lambda,\mu,\lambda}{n-1-k}(4k-2\beta)\,.
\end{aligned}
$$

Further essential simplifications seem to be possible only for special values of α, in particular the standard case $\alpha=\beta=\frac{1}{2}$.

Note that the above computation was equivalent to the use of the Lagrange inversion formula.

Odd number of steps. Let us start with the easy case of paths bounded from above by $2h+1$. It means that the modified path is bounded from above by h, and an additional step at the end cannot bring the path to a level larger than $2h+1$. Thus

$$M_{\mathsf{odd}}^{[\le 2h+1]}=\sum_{i\le h}\varphi_i$$

is also the generating function of paths of length $2n+1$, bounded from above by $2h+1$.

Now, we consider paths bounded from above by $2h$. Their enumeration was achieved as

$$\sum_{i\le h}\varphi_i$$

(with φ different from before). Now an additional step can be done in almost all situations. The only exception is, that, when the path has ended on level h (original path on level $2h$), an up-step is forbidden. Thus, the quantity $\alpha\varphi_h$ has to be subtracted, leading to

$$M_{\mathsf{odd}}^{[\le 2h]}=\sum_{i\le h}\varphi_i-\alpha\varphi_h.$$

Thus

$$M_{\mathsf{odd}}^{[\le 2h+1]}=M^{[\le 2h+1]}(z)=\frac{(\lambda+\mu v+\lambda v^2)(1-v^{h+1})}{\lambda(1-v)^2}.$$

Further,

$$M_{\mathsf{odd}}^{[>2h+1]}=M^{[>2h+1]}(z)=\frac{(\lambda+\mu v+\lambda v^2)v^{h+1}}{\lambda(1-v)^2}.$$

For the other instance,

$$M_{\mathsf{odd}}^{[\le 2h]} = M^{[\le 2h]}(z) - \alpha v^h \frac{\lambda+\mu v+\lambda v^2}{\lambda}\frac{1}{1+\frac{\alpha}{\beta}v}.$$

Therefore

$$\begin{aligned} M_{\mathsf{odd}}^{[>2h]} = M^{[>2h]}(z) &= \frac{(\lambda+\mu v+\lambda v^2)v^{h+1}}{\lambda(\beta+\alpha v)(1-v)^2} + \frac{(\lambda+\mu v+\lambda v^2)v^h}{\beta+\alpha v} \\ &= \frac{(\lambda+\mu v+\lambda v^2)^2 v^h}{\lambda(\beta+\alpha v)(1-v)^2}. \end{aligned}$$

From this we find immediately the generating function of the expectations:

$$\begin{aligned} E_{\mathsf{odd}}(z) &= \sum_{h\ge 0} M_{\mathsf{odd}}^{[>h]}(z) = \sum_{h\ge 0} M_{\mathsf{odd}}^{[>2h]}(z) + \sum_{h\ge 0} M_{\mathsf{odd}}^{[>2h+1]}(z) \\ &= \frac{\lambda+\mu v+\lambda v^2}{\lambda(1-v)^2}\left[\frac{\lambda+\mu v+\lambda v^2}{\beta+\alpha v}+v\right]\frac{1}{1-v} \\ &= \frac{\lambda+\mu v+\lambda v^2}{\lambda(1-v)^3}\left[\alpha+\beta v+v\right] = \frac{\lambda+\mu v+\lambda v^2}{\lambda(1-v)^3}\left[\alpha-\alpha v+2v\right] \\ &= \frac{\lambda+\mu v+\lambda v^2}{\lambda}\left[\frac{2v}{(1-v)^3}+\frac{\alpha}{(1-v)^2}\right]. \end{aligned}$$

The coefficients of this generating function can again be written as a linear combination of trinomial coefficients:

$$[z^n]E_{\mathsf{odd}}(z) = \mathbb{E}\mathsf{Max}_{2n+1} = \alpha\binom{n;\lambda,\mu,\lambda}{n} + \sum_{k=1}^{n}\binom{n;\lambda,\mu,\lambda}{n-k}(4k-2\beta).$$

Asymptotic evaluation. Let us now move to asymptotics. In order to apply singularity analysis of generating functions [4, 5], we need the local expansion around $z=1$. Since

$$\begin{aligned} v &\sim 1-\sqrt{\frac{1-z}{\lambda}}+\frac{1-z}{2\lambda}-\frac{4\lambda+1}{8}\left(\frac{1-z}{\lambda}\right)^{3/2}+\cdots \\ &\leftrightarrow 1-z \sim \lambda(1-v)^2+\lambda(1-v)^3+\lambda(1-\lambda)(1-v)^4+\cdots \end{aligned}$$

we get

$$\begin{aligned} E(z) &= \frac{2}{\lambda(1-v)^3}-\frac{4-\alpha}{\lambda(1-v)^2}+\frac{2-\alpha^2}{\lambda(1-v)}-1 \\ &\sim \frac{2\sqrt{\lambda}}{(1-z)^{3/2}}-\frac{1-\alpha}{1-z}+\frac{1-8\lambda}{4\sqrt{\lambda}(1-z)^{1/2}}+\frac{1-2\alpha}{2} \\ &\qquad + \frac{1-16\lambda^2}{64\lambda^{3/2}}(1-z)^{1/2}+O\left((1-z)^{3/2}\right). \end{aligned}$$

The same procedure for $E_{\mathsf{odd}}(z)$ furnishes

$$E_{\mathsf{odd}}(z) = \frac{2}{\lambda(1-v)^3}-\frac{4-\alpha}{\lambda(1-v)^2}+\frac{2+\alpha-2\alpha^2}{\lambda(1-v)}+\alpha-2$$

$$\sim \frac{2\sqrt{\lambda}}{(1-z)^{3/2}} - \frac{1-\alpha}{1-z} + \frac{1-4\lambda}{4\sqrt{\lambda}(1-z)^{1/2}} + \frac{1-8\lambda+16\lambda^2}{64\lambda^{3/2}}(1-z)^{1/2} + O\left((1-z)^{3/2}\right).$$

These local expansions of $E(z)$ and $E_{\mathsf{odd}}(z)$ could be translated into the asymptotic expansions of the coefficients of z^n, corresponding to the asymptotic expected maximum after $2n$ and $2n+1$ steps, respectively, of the original path. But, observing that the generating function $G(x)$ of the expected maximum of the original path is $E(x^2) + xE_{\mathsf{odd}}(x^2)$, we can even derive explicit expressions for the expected maximum after n steps of the original path, as $n \to \infty$. Since $E(z)$ and $E_{\mathsf{odd}}(z)$ have their singularities at $z = 1$, $G(x)$ has its singularities at $x = 1$ and $x = -1$. So we only have to expand $G(x) = E(x^2) + xE_{\mathsf{odd}}(x^2)$ around $x = 1$ and $x = -1$, add up, and translate (by applying the transfer theorem from singularity analysis of generating functions, see [4]) into the asymptotic expansion of the coefficient of x^n. Proceeding in that way we get

$$G(x) = \frac{2\sqrt{\lambda}}{\sqrt{2}}(1-x)^{-3/2} - (1-\alpha)(1-x)^{-1} + \frac{1-5\lambda}{2\sqrt{2\lambda}}(1-x)^{-1/2} + \frac{1-2\alpha}{2} - \frac{1-2\lambda-7\lambda^2}{16\sqrt{2}\lambda^{3/2}}(1-x)^{1/2} + O\big((1-x)^{3/2}\big), \qquad \text{as } x \to 1,$$

and

$$G(x) = \frac{-\alpha}{2} + \frac{\lambda^2}{2\sqrt{2}\lambda^{3/2}}(1+x)^{1/2} - \frac{1-\alpha}{4}(1+x) + O\big((1+x)^{3/2}\big), \qquad \text{as } x \to -1.$$

Hence the expected maximum $\mathbb{E}\mathsf{Max}_n$ of the original path after n steps is

$$\mathbb{E}\mathsf{Max}_n = [x^n]\,G(x) = \frac{4\sqrt{\lambda}\sqrt{n}}{\sqrt{2\pi}} - (1-\alpha) + \frac{1-2\lambda}{2\sqrt{\lambda}\sqrt{2\pi}\sqrt{n}} - \frac{4\lambda^2+4\lambda-1}{32\lambda^{3/2}\sqrt{2\pi}\,n^{3/2}} - \frac{(-1)^n\sqrt{\lambda}}{4\sqrt{2\pi}\,n^{3/2}} + O\left(n^{-5/2}\right). \tag{2}$$

This refines a recent result of Böhm and Hornik [1]. Our additional terms show that the expected maximum actually is different for paths of even and odd length; these differences occur only for lower order terms.

We summarize our findings.

Theorem 1. *The generating functions $E(z)$ resp. $E_{\text{odd}}(z)$ of the maximum of the two-periodic paths with weights α and β after $2n$ resp. $2n+1$ random steps is given by*

$$E(z) = \frac{v(\alpha+\beta v)(1+\beta+\alpha v)}{\lambda(1-v)^3},$$

$$E_{\text{odd}}(z) = \frac{(\lambda+\mu v+\lambda v^2)(\alpha+\beta v+v)}{\lambda(1-v)^3},$$

with $z = \dfrac{v}{\lambda+\mu v+\lambda v^2}$*. The expected value of the maximum after* n *random steps is given by*

$$\mathbb{E}\mathsf{Max}_n \sim \frac{4\sqrt{\lambda}\sqrt{n}}{\sqrt{2\pi}},$$

with more terms provided in (2).

3. The two-sided maximum

In this section we consider the two-sided maximum of a walk $(0, s_0)$, $(1, s_1)$, $\dots$, (n, s_n), defined to be $\max_{0\le i\le n}\{|s_i|\}$. We have to find (probability) generating functions for the walks that are bounded by h. As before, it has to be distinguished whether h is even or odd. Let us start with the easy case that the walk lives in the strip $-(2h+1) \le \dots \le 2h+1$. Then the computations from the previous section can be used, with $h = k$. From (1) we find

$$T^{[\le 2h+1]} = \sum_{i=-h}^{h}\varphi_i = \frac{\lambda+\mu v+\lambda v^2}{\lambda}\frac{(1-v^{h+1})^2}{(1-v)^2(1+v^{2h+2})}$$

and

$$T^{[>2h+1]} = \frac{\lambda+\mu v+\lambda v^2}{\lambda(1-v)^2}\frac{2v^{h+1}}{1+v^{2h+2}}.$$

Now we move to the instance $-2h \le \dots \le 2h$. Here, we have to do some independent computations. We need some more determinants. Let

$$b_i^* = \begin{vmatrix} 1-\mu z & -\lambda z & & & \\ -\lambda z & 1-\mu z & -\lambda z & & \\ & & \ddots & & \\ & & -\lambda z & 1-\mu z & -\lambda z \\ & & & -\lambda z & 1-\alpha^2 z \end{vmatrix}$$

with i rows. Then

$$b_i^* = b_i\big|_{\alpha\leftrightarrow\beta} = \frac{1}{1-v^2}\Big(\frac{\lambda}{\lambda+\mu v+\lambda v^2}\Big)^i\Big[1+\frac{\beta}{\alpha}v-\frac{\beta}{\alpha}v^{2i+1}-v^{2i+2}\Big].$$

Let

$$c_i = \begin{vmatrix} 1-\beta^2 z & -\lambda z & & & \\ -\lambda z & 1-\mu z & -\lambda z & & \\ & & \ddots & & \\ & & -\lambda z & 1-\mu z & -\lambda z \\ & & & -\lambda z & 1-\alpha^2 z \end{vmatrix}$$

with i rows. Then

$$\begin{aligned} c_i &= (1-\alpha^2 z)b_{i-1} - \lambda^2 z^2 b_{i-2} \\ &= \frac{1-v^{2i}}{1-v^2}\Big(\frac{\lambda}{\lambda+\mu v+\lambda v^2}\Big)^{i-1}. \end{aligned}$$

With these determinants, the generating functions, leading to level i with $-h \le i \le h$, can be expressed:

$$\varphi_i = \frac{(\lambda z)^i b_{h-i} b_h^*}{c_{2h+1}}, \quad i = 0, \dots, h,$$
$$\varphi_{-i} = \frac{(\lambda z)^i b_{h-i}^* b_h}{c_{2h+1}}, \quad i = 0, \dots, h.$$

This leads to

$$\varphi_i = v^i \frac{\left[1 + \frac{\alpha}{\beta} v - \frac{\alpha}{\beta} v^{2(h-i)+1} - v^{2(h-i)+2}\right]\left[1 + \frac{\beta}{\alpha} v - \frac{\beta}{\alpha} v^{2h+1} - v^{2h+2}\right]}{(1 - v^{2(2h+1)})(1 - v^2)},$$
$$\varphi_{-i} = \varphi_i\Big|_{\alpha \leftrightarrow \beta},$$

and

$$\sum_{i=0}^{h} \varphi_i = \frac{\left[1 + \frac{\beta}{\alpha} v - \frac{\beta}{\alpha} v^{2h+1} - v^{2h+2}\right]}{(1 - v^{2(2h+1)})(1 - v^2)} \sum_{i=0}^{h} v^i \left[1 + \frac{\alpha}{\beta} v - \frac{\alpha}{\beta} v^{2(h-i)+1} - v^{2(h-i)+2}\right]$$
$$= \frac{[(1 + \frac{\beta}{\alpha} v) - v^{2h+1}(\frac{\beta}{\alpha} + v)][(1 + \frac{\alpha}{\beta} v) - v^{h+1}(\frac{\alpha}{\beta} + v)]}{(1 - v^{2(2h+1)})(1 - v)(1 - v^2)}(1 - v^{h+1}).$$

Likewise,

$$\sum_{i=0}^{h} \varphi_{-i} = \frac{[(1 + \frac{\alpha}{\beta} v) - v^{2h+1}(\frac{\alpha}{\beta} + v)][(1 + \frac{\beta}{\alpha} v) - v^{h+1}(\frac{\beta}{\alpha} + v)]}{(1 - v^{2(2h+1)})(1 - v)(1 - v^2)}(1 - v^{h+1}).$$

Then

$$T^{[\le 2h]} = \sum_{i=-h}^{h} \varphi_i = \sum_{i=0}^{h} \varphi_i + \sum_{i=0}^{h} \varphi_{-i} - \varphi_0$$
$$= \frac{(\lambda + \mu v + \lambda v^2)(1 - v^{4h+2}) - (1 + v) v^{h+1}(1 - v^{2h+1})}{\lambda(1 - v^{4h+2})(1 - v)^2}$$
$$= \frac{(\lambda + \mu v + \lambda v^2)}{\lambda(1 - v)^2} - \frac{(1 + v) v^{h+1}}{\lambda(1 + v^{2h+1})(1 - v)^2}.$$

Therefore

$$T^{[>2h]} = \frac{(1 + v) v^{h+1}}{\lambda(1 + v^{2h+1})(1 - v)^2}.$$

This leads to the generating function of the expected values of the two-sided maximum:

$$E(z) = \sum_{h \ge 0} T^{[>h]} = \sum_{h \ge 0} T^{[>2h]} + \sum_{h \ge 0} T^{[>2h+1]}$$
$$= \frac{(1 + v)}{\lambda(1 - v)^2} \sum_{h \ge 0} \frac{v^{h+1}}{1 + v^{2h+1}} + \frac{2(\lambda + \mu v + \lambda v^2)}{\lambda(1 - v)^2} \sum_{h \ge 1} \frac{v^h}{1 + v^{2h}}.$$

Odd number of steps. This will now be quite similar to before. There are exceptional cases for even $2h$, which means h for the modified path, and an up-step at level h and a down-step at level $-h$ must be excluded. This leads to slightly modified generating functions.

$$T^{[\le 2h+1]}_{\mathsf{odd}} = T^{[\le 2h+1]},$$
$$T^{[\le 2h]}_{\mathsf{odd}} = T^{[\le 2h]} - \alpha\varphi_h - \beta\varphi_{-h}.$$

Therefore

$$T^{[\le 2h+1]}_{\mathsf{odd}} = T^{[\le 2h+1]} = \frac{\lambda+\mu v+\lambda v^2}{\lambda}\,\frac{(1-v^{h+1})^2}{(1-v)^2(1+v^{2h+2})}$$

and also

$$T^{[>2h+1]}_{\mathsf{odd}} = T^{[>2h+1]} = \frac{\lambda+\mu v+\lambda v^2}{\lambda(1-v)^2}\,\frac{2v^{h+1}}{1+v^{2h+2}}\,,$$

$$T^{[\le 2h]} = \frac{\lambda+\mu v+\lambda v^2}{\lambda(1-v)^2} - \frac{(1+v)v^{h+1}}{\lambda(1+v^{2h+1})(1-v)^2}\,.$$

We have

$$\begin{aligned}
\varphi_h &= v^h\,\frac{\left[1+\frac{\alpha}{\beta}v-\frac{\alpha}{\beta}v-v^2\right]\left[1+\frac{\beta}{\alpha}v-\frac{\beta}{\alpha}v^{2h+1}-v^{2h+2}\right]}{(1-v^{2(2h+1)})(1-v^2)}\\
&= v^h\,\frac{(1-v^2)\left[\left(1-v^{2h+2}\right)+\frac{\beta}{\alpha}v\left(1-v^{2h}\right)\right]}{(1-v^{2(2h+1)})(1-v^2)}\\
&= v^h\,\frac{\left(1-v^{2h+2}\right)+\frac{\beta}{\alpha}v\left(1-v^{2h}\right)}{1-v^{2(2h+1)}}
\end{aligned}$$

and further

$$\begin{aligned}
\alpha\varphi_h+\beta\varphi_{-h} &= \alpha v^h\,\frac{\left(1-v^{2h+2}\right)+\frac{\beta}{\alpha}v\left(1-v^{2h}\right)}{1-v^{2(2h+1)}} + \beta v^h\,\frac{\left(1-v^{2h+2}\right)+\frac{\alpha}{\beta}v\left(1-v^{2h}\right)}{1-v^{2(2h+1)}}\\
&= \frac{v^h}{1-v^{2(2h+1)}}\left[(\alpha+\beta)(1-v^{2h+2})+(\alpha+\beta)v(1-v^{2h})\right]\\
&= \frac{v^h}{1-v^{2(2h+1)}}\left[(1+v)-v^{2h+1}(1+v)\right]\\
&= \frac{(1+v)v^h}{1+v^{2h+1}}.
\end{aligned}$$

Thus

$$\begin{aligned}
T^{[\le 2h]}_{\mathsf{odd}} &= \frac{\lambda+\mu v+\lambda v^2}{\lambda(1-v)^2} - \frac{(1+v)v^{h+1}}{\lambda(1+v^{2h+1})(1-v)^2} - \frac{(1+v)v^h}{1+v^{2h+1}}\\
&= \frac{\lambda+\mu v+\lambda v^2}{\lambda(1-v)^2} - \frac{(1+v)v^h(\lambda+\mu v+\lambda v^2)}{\lambda(1+v^{2h+1})(1-v)^2}\\
&= \frac{\lambda+\mu v+\lambda v^2}{\lambda(1-v)^2}\left[1-\frac{(1+v)v^h}{1+v^{2h+1}}\right]
\end{aligned}$$

$$= \frac{\lambda+\mu v+\lambda v^2}{\lambda(1-v)^2}\cdot\frac{1-v^h-v^{h+1}+v^{2h+1}}{1+v^{2h+1}}$$
$$= \frac{(\lambda+\mu v+\lambda v^2)(1-v^h)(1-v^{h+1})}{\lambda(1-v)^2(1+v^{2h+1})}.$$

This leads to

$$T_{\mathsf{odd}}^{[>2h+1]} = \frac{2(\lambda+\mu v+\lambda v^2)v^{h+1}}{\lambda(1-v)^2(1+v^{2h+2})} = T^{[>2h+1]},$$
$$T_{\mathsf{odd}}^{[>2h]} = \frac{(\lambda+\mu v+\lambda v^2)(1+v)v^h}{\lambda(1-v)^2(1+v^{2h+1})}.$$

$$E_{\mathsf{odd}}(z) = \sum_{h\ge0} T_{\mathsf{odd}}^{[>h]} = \sum_{h\ge0} T_{\mathsf{odd}}^{[>2h]} + \sum_{h\ge0} T_{\mathsf{odd}}^{[>2h+1]}$$
$$= \frac{(\lambda+\mu v+\lambda v^2)(1+v)}{\lambda(1-v)^2}\sum_{h\ge0}\frac{v^h}{1+v^{2h+1}} + \frac{2(\lambda+\mu v+\lambda v^2)}{\lambda(1-v)^2}\sum_{h\ge1}\frac{v^h}{1+v^{2h}}.$$

Asymptotic evaluation. Let us now move to asymptotics. Regarding the series (harmonic sums) in the expressions of $E(z)$ and $E_{\mathsf{odd}}(z)$ this is more difficult than before. To deal with them let us introduce

$$\sigma_1 = \sum_{h\ge0}\frac{v^{h+\frac12}}{1+v^{2h+1}}, \qquad \sigma_2 = \sum_{h\ge1}\frac{v^h}{1+v^{2h}}.$$

Using σ_1, σ_2 we get

$$E(z) = \frac{1}{\lambda(1-v)^2}\left[(1+v)\sqrt{v}\sigma_1 + 2(\lambda+\mu v+\lambda v^2)\sigma_2\right],$$
$$E_{\mathsf{odd}}(z) = \frac{\lambda+\mu v+\lambda v^2}{\lambda(1-v)^2}\left[\frac{1+v}{\sqrt{v}}\sigma_1 + 2\sigma_2\right].$$

Note that

$$\sigma_1 = \sum_{h\ge0}\sum_{k\ge0}(-1)^k v^{(h+\frac12)(2k+1)} \quad\text{and}\quad \sigma_2 = \sum_{h\ge1}\sum_{k\ge0}(-1)^k v^{h(2k+1)}.$$

For the further study, one sets $v = e^{-t}$ and performs the Mellin transform [3]:

$$\mathcal{M}\sum_{h\ge0}\sum_{k\ge0}(-1)^k e^{-t(h+\frac12)(2k+1)} = \Gamma(s)\sum_{h\ge0}\sum_{k\ge0}(-1)^k(h+\tfrac12)^{-s}(2k+1)^{-s}$$
$$= \Gamma(s)\sum_{h\ge0}(h+\tfrac12)^{-s}\sum_{k\ge0}(-1)^k(2k+1)^{-s}$$
$$= \Gamma(s)\zeta(s,\tfrac12)\left[\sum_{k\ge0}(4k+1)^{-s} - \sum_{k\ge0}(4k+3)^{-s}\right]$$
$$= 4^{-s}\Gamma(s)\zeta(s,\tfrac12)\left[\zeta(s,\tfrac14)-\zeta(s,\tfrac34)\right],$$

$$\begin{aligned}\mathcal{M}\sum_{h\ge1}\sum_{k\ge0}(-1)^k e^{-th(2k+1)} &= \Gamma(s)\sum_{h\ge1}\sum_{k\ge0}(-1)^k h^{-s}(2k+1)^{-s}\\ &= 4^{-s}\Gamma(s)\zeta(s)\big[\zeta(s,\tfrac14)-\zeta(s,\tfrac34)\big].\end{aligned}$$

Here, we used the Hurwitz' zeta function

$$\zeta(s,a)=\sum_{n\ge0}\frac{1}{(n+a)^s}.$$

A reference for many of its properties is the classic book [8].

The method consists in applying the inverse Mellin transform, which is given as a contour integral, and evaluating it (asymptotically) by computing residues of

$$4^{-s}\Gamma(s)\zeta(s,\tfrac12)\big[\zeta(s,\tfrac14)-\zeta(s,\tfrac34)\big]t^{-s}$$

and

$$4^{-s}\Gamma(s)\zeta(s)\big[\zeta(s,\tfrac14)-\zeta(s,\tfrac34)\big]t^{-s}.$$

The contour of integration is a vertical line that is to the right of possible singularities. We can take, for instance $\Re s=2$. The general strategy is to move the line of integration to the left and collect residues. The error comes from the new vertical line, i. e., if the line is moved to $\Re c$, then we get an error $O\left(t^{-c}\right)$. Sometimes, there are also logarithmic terms involved in the error term, but not in our instance. See [3] for details.

The function $\beta(s)=\zeta(s,\frac14)-\zeta(s,\frac34)$ is an entire function, since the poles cancel out. Thus, we encounter a simple pole at $s=1$, originating from $\zeta(s)$ and $\zeta(s,\frac12)$, respectively. The respective residues are $\frac{\pi}{4t}$ in both instances.

The Gamma function has simple poles at $0,-1,-2,\dots$ But, since $\zeta(s,\frac12)=0$ for $s=0,-2,-4,\dots$, the pertaining poles of the Gamma function are compensated by the zeroes of $\zeta(s,\frac12)$ and σ_1 has no residues at these points. The situation is quite similar for σ_2, since $\zeta(s)=0$ for $s=-2,-4,-6,\dots$ Only the pole in $s=0$ survives. The pertaining residue of σ_2 is $-1/4$, since $\zeta(0)=-1/2$ and $\beta(0)=1/2$.

Also the poles at $s=-1,-3,\dots$ are compensated because $\beta(-(2m-1))=0$ for $m=1,2,3,\dots$. This can easily be shown by using the well-known relation

$$\zeta(-n,x)=-\frac{B_{n+1}(x)}{n+1}.$$

But the Bernoulli polynomials $B_n(x)$ are symmetric around $x=\frac12$,

$$B_n(\tfrac12+x)=(-1)^nB_n(\tfrac12-x),$$

which entails $\zeta(-(2m-1),1/4)-\zeta(-(2m-1),3/4)=0$. So taking all residues into acount we finally obtain

$$\sigma_1=\frac{\pi}{4t}+O\left(t^M\right)\quad\text{and}\quad\sigma_2=\frac{\pi}{4t}-\frac14+O\left(t^M\right)$$

for arbitrarily large M, and we may replace

$$\sigma_1\sim-\frac{\pi}{4\,\ln v}\quad\text{and}\quad\sigma_2\sim-\frac{\pi}{4\,\ln v}-\frac14,$$

after reverting back to v, and the final error terms come from the other factors and are related to the number of terms that we compute.

As in the one-sided case $E(z)$ and $E_{\mathsf{odd}}(z)$ have their singularities at $v=1$, so we may proceed as in the preceding section. Expanding $E(z)$ around $v=1$ and translating to $z=1$ yields (these computations have been done with a computer)

$$\begin{aligned} E(z) &= \frac{\pi}{\lambda(1-v)^3} - \frac{1+3\pi}{2\lambda(1-v)^2} + \frac{24\pi\lambda+23\pi+24}{48\lambda(1-v)} + \frac{-24\pi\lambda-48\lambda+\pi}{96\lambda} \\ &\quad - \frac{\pi(480\lambda-71)(1-v)}{11520\lambda} - \frac{\pi(480\lambda-71)(1-v)^2}{23040\lambda} + O\left((1-v)^3\right) \\ &= \frac{\pi\sqrt{\lambda}}{(1-z)^{3/2}} - \frac{1}{2(1-z)} - \frac{\pi(48\lambda-5)}{48\sqrt{\lambda}\sqrt{1-z}} \\ &\quad + \frac{\pi\left(1440\lambda^2-360\lambda-49\right)\sqrt{1-z}}{11520\lambda^{3/2}} + O\left((1-z)^{3/2}\right). \end{aligned}$$

The same procedure for $E_{\mathsf{odd}}(z)$ furnishes

$$\begin{aligned} E_{\mathsf{odd}}(z) &= \frac{\pi}{\lambda(1-v)^3} - \frac{1+3\pi}{2\lambda(1-v)^2} + \frac{48\pi\lambda+23\pi+24}{48\lambda(1-v)} + \frac{-48\pi\lambda-48\lambda+\pi}{96\lambda} \\ &\quad - \frac{(240\lambda-71)\pi(1-v)}{11520\lambda} - \frac{(240\lambda-71)\pi(1-v)^2}{23040\lambda} + O\left((1-v)^3\right) \\ &= \frac{\sqrt{\lambda}\pi}{(1-z)^{3/2}} - \frac{1}{2(1-z)} + \frac{5\pi-24\lambda\pi}{48\sqrt{\lambda}\sqrt{1-z}} \\ &\quad + \frac{\left(1440\pi\lambda^2-600\pi\lambda+49\pi\right)\sqrt{1-z}}{11520\lambda^{3/2}} + O\left((1-z)^{3/2}\right). \end{aligned}$$

The local expansions of $G(x)=E(x^2)+xE_{\mathsf{odd}}(x^2)$ around $x=1$ and $x=-1$ are

$$\begin{aligned} G(x) &= \frac{\pi\sqrt{\lambda}}{\sqrt{2}}(1-x)^{-3/2} - \frac{1}{2}(1-x)^{-1} - \frac{5\pi(6\lambda-1)}{24\sqrt{2}\sqrt{\lambda}}(1-x)^{-1/2} \\ &\quad + \frac{\pi(630\lambda^2-30\lambda-49)}{2880\sqrt{2}\,\lambda^{3/2}}(1-x)^{1/2} + O\big((1-x)^{3/2}\big), \qquad \text{as } x\to 1, \end{aligned}$$

and

$$G(x) = -\frac{1}{4} + \frac{\pi(4\lambda-1)}{16\sqrt{2}\sqrt{\lambda}}(1+x)^{1/2} - \frac{1+x}{8} + O\big((1+x)^{3/2}\big), \qquad \text{as } x\to -1.$$

Hence the expected two-sided maximum $\mathbb{E}\mathsf{Max}_n$ of the original path after n steps is

$$\begin{aligned} \mathbb{E}\mathsf{Max}_n &= [x^n]G(x) \\ &= \sqrt{2\pi}\sqrt{\lambda}\sqrt{n} - \frac{1}{2} + \frac{\sqrt{2\pi}\,(5-12\lambda)}{48\sqrt{\lambda}\sqrt{n}} + \frac{\sqrt{2\pi}\,(49-120\lambda-360\lambda^2)}{11520\,\lambda^{3/2}\,n^{3/2}} \\ &\quad + (-1)^n\frac{\sqrt{2\pi}(1-4\lambda)}{64\sqrt{\lambda}\,n^{3/2}} + O\left(n^{-5/2}\right). \end{aligned} \tag{3}$$

We summarize our findings in the following theorem.

Theorem 2. *The generating functions $E(z)$ resp. $E_{\mathrm{odd}}(z)$ of the two-sided maximum of the two-periodic paths with weights α and β after $2n$ resp. $2n+1$ random steps is given by*

$$E(z)=\frac{1}{\lambda(1-v)^2}\left[(1+v)\sum_{h\ge0}\frac{v^{h+1}}{1+v^{2h+1}}+2(\lambda+\mu v+\lambda v^2)\sum_{h\ge1}\frac{v^h}{1+v^{2h}}\right],$$

$$E_{\mathrm{odd}}(z)=\frac{(\lambda+\mu v+\lambda v^2)}{\lambda(1-v)^2}\left[(1+v)\sum_{h\ge0}\frac{v^h}{1+v^{2h+1}}+2\sum_{h\ge1}\frac{v^h}{1+v^{2h}}\right],$$

with $z=\dfrac{v}{\lambda+\mu v+\lambda v^2}$. The expected value of the two-sided maximum after n random steps is given by

$$\mathbb{E}\mathsf{Max}_n\sim\sqrt{2\pi}\sqrt{\lambda}\sqrt{n},$$

with more terms provided in (3).

4. Conclusion

Our combinatorial analysis of two-periodic random walks followed the scheme: reduction to the study of Motzkin paths, where n steps mean $2n$ steps in the original walks. For walks of odd length a small modification is necessary. The generating functions of interest were computed by solving a linear system with Cramer's rule. A suitable substitution made the expressions manageable. Asymptotics follow from singularity analysis of generating functions, in one case with a Mellin transform argument as a preliminary step. Computer algebra was very helpful, but human interaction was essential.

We hope that we succeeded to convince the reader of the advantages of our method. Only two parameters were studied: one- and two-sided maximum. We plan to come back to the subject in other publications by investigating other parameters, and also different random walk models, i. e., walks that must return to the x-axis after n steps, and non-negative walks with or without prescribed endpoints.

References

[1] Walter Böhm and Kurt Hornik. On two-periodic random walks with boundaries. *Stoch. Models*, 26(2):165–194, 2010.

[2] Louis Comtet. *Advanced combinatorics*. D. Reidel Publishing Co., Dordrecht, enlarged edition, 1974. The art of finite and infinite expansions.

[3] Philippe Flajolet, Xavier Gourdon, and Philippe Dumas. Mellin transforms and asymptotics: Harmonic sums. *Theoretical Computer Science*, 144:3–58, 1995.

[4] Philippe Flajolet and Andrew Odlyzko. Singularity analysis of generating functions. *SIAM Journal on Discrete Mathematics*, 3:216–240, 1990.

[5] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009.

[6] Wolfgang Panny and Helmut Prodinger. The expected height of paths for several notions of height. *Studia Sci. Math. Hungar.*, 20(1-4):119–132, 1985.

[7] Helmut Prodinger. Analytic methods, 84 pages. *In: Handbook of Enumerative Combinatorics, M. Bona, Ed.*, 2015.

[8] Edmund Taylor Whittaker and George Neville Watson. *A course of modern analysis*. Cambridge University Press, Cambridge, 4th edition, 1927. reprinted 1996.

Department of Information Systems and Operations, Vienna University of Economics and Business, 1020 Vienna, Austria

E-mail address: wolfgang.panny@wu.ac.at

Department of Mathematics, University of Stellenbosch 7602 Stellenbosch South Africa

E-mail address: hproding@sun.ac.za