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The number of restricted lattice paths revisited

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Abstract. Ilić and Ilić have recently discussed lattice paths starting and ending at the x-axis which are bounded by two horizontal lines. We establish a link of this to an old paper by Panny and Prodinger where this was already treated.

In [1] the number of random walks from $(0,0)$ to $(2n,0)$ with up-steps and down-steps of one unit each was discussed, under the condition that the walk (path) never touches the line $-h$ and k. Here, we want to shed additional light on this, by pointing out that this appeared essentially already in our 1985 paper [2]. Since all this is not complicated, we review the essential steps here. We allow the path to touch $-h$ and k, but not $-h-1$ and $k+1$. Further, let $\psi_i(z)$ be the generating function, for $-h \le i \le k$, of paths in the sense just described that lead to level i. Eventually, we are interested in $\psi_0(z)$.

The following system of linear equations is self-explanatory (and discussed at length in [2]):

$$\begin{bmatrix} 1 & -z & 0 & \dots & & \\ -z & 1 & -z & 0 & \dots & \\ 0 & -z & 1 & -z & 0 & \dots \\ & & & \dots & & \\ & & & -z & 1 & -z \\ & & & & -z & 1 \end{bmatrix} \begin{bmatrix} \psi_{-h}(z) \\ \vdots \\ \psi_0(z) \\ \vdots \\ \psi_k(z) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}.$$

We use Cramer's rule to solve this:

$$\psi_0(z) = \frac{a_{h-1}a_{k-1}}{a_{h+k}},$$

where a_i is the determinant of the square matrix with $i+1$ rows and columns. Since a_i satisfies a recursion of second order, it is easy to get

$$a_i = \frac{1}{1-v^2}\frac{1-v^{2i+4}}{(1+v^2)^{i+1}},$$

where the substitution $z = v/(1+v^2)$ was used for convenience.

We want to make the parameters h and k explicit and define

$$f_{h,k}(z) = \psi_0(z).$$

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The example with lines -2 and 5 corresponds to our $f_{1,4}(z)$. We compute (with Maple):

$$f_{1,4}(z) = 1 + 2z^2 + 5z^4 + 14z^6 + 42z^8 + 131z^{10} + 417z^{12} + 1341z^{14} + 4334z^{16} + 14041z^{18} \\ + 45542z^{20} + 147798z^{22} + 479779z^{24} + 1557649z^{26} + 5057369z^{28} + \dots,$$

in agreement with [1].

Since (by Cauchy's integral formula or Lagrange inversion)

$$\begin{aligned}[z^{2n}]f_{h,k}(z) &= [v^{2n}](1+v^2)^{2n}\frac{(1-v^{2h+2})(1-v^{2k+2})}{(1-v^{2h+2k+4})}\\ &= [v^n](1+v)^{2n}\frac{(1-v^{h+1})(1-v^{k+1})}{(1-v^{h+k+2})}\\ &= \sum_{j\ge0}\Bigg[\binom{2n}{n-j(h+k+2)}-\binom{2n}{n-j(h+k+2)-h-1}\\ &\qquad -\binom{2n}{n-j(h+k+2)-k-1}+\binom{2n}{n-(j+1)(h+k+2)}\Bigg],\end{aligned}$$

we have even an explicit formula. For $h = 1$ and $k = 4$, this gives the sequence

$$1, 2, 5, 14, 42, 131, 417, 1341, 4334, 14041, 45542, 147798, 479779, 1557649, 5057369, \dots,$$

as expected.

References

[1] A. Ilić and A. Ilić, On the number of restricted Dyck paths, *Filomat*, 25 (2011), 191–201.
[2] W. Panny and H. Prodinger, The expected height of paths for several notions of height, *Stud. Sci. Math. Hung.*, 20 (1985), 119–132.