THE q-PILBERT MATRIX

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Abstract. A generalized Filbert matrix is introduced, sharing properties of the Hilbert matrix and Fibonacci numbers. Explicit formulæ are derived for the LU-decomposition and their inverses, as well as the Cholesky decomposition. The approach is to use q-analysis and to leave the justification of the necessary identities to the q-version of Zeilberger's celebrated algorithm.

1. Introduction

The Filbert (=*F*ibonacci-H*ilbert*) matrix $H_n = \left(\check{h}_{ij}\right)_{i,j=1}^{n}$ is defined by $\check{h}_{ij} = \frac{1}{F_{i+j-1}}$ as an analogue of the Hilbert matrix where F_n is the nth Fibonacci number. It has been defined and studied by Richardson [6].

In [1], Kilic and Prodinger studied the generalized matrix with entries $\frac{1}{F_{i+j+r}}$, where $r \geq -1$ is an integer parameter. They gave its LU factorization and, using this, computed its determinant and inverse. Also the Cholesky factorization was derived. After this generalization, Prodinger [5] defined a new generalization of the generalized Filbert matrix by introducing 3 additional parameters. Again, explicit formulæ for the LU-decomposition, their inverses, and the Cholesky factorization were derived.

In this paper we will consider a further generalization of the generalized Filbert Matrix $\mathcal{F}$ with entries $\frac{1}{F_{i+j+r}}$, where $r \geq -1$ is an integer parameter. We define the matrix $\mathcal{Q}$ with entries h_{ij} as follows

$$h_{ij} = \frac{1}{F_{i+j+r}F_{i+j+r+1}\dots F_{i+j+r+k-1}},$$

where $r \geq -1$ is an integer parameter and $k \geq 0$ is an integer parameter.

When $k = 1$, we get the generalized Filbert Matrix $\mathcal{F}$, as studied before.

In this paper we shall derive explicit formulæ for the LU-decomposition and their inverses. Similarly to the results of [1], the size of the matrix does not really matter, and we can think about an infinite matrix $\mathcal{Q}$ and restrict it whenever necessary to the first n rows resp. columns and write $\mathcal{Q}_n$. All the identities we will obtain hold for general q, and results about Fibonacci numbers come out as corollaries for the special choice of q. The entries of the inverse matrix $\mathcal{Q}_n^{-1}$ are not closed form expressions, as in our previous paper, but can only be given as a (simple) sum. We also provide the Cholesky decomposition.

Our approach will be as follows. We will use the Binet form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q},$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$.

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Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

Considering the definitions of the matrix $\mathcal{Q}$ and q-Pochhammer symbol, we rewrite the matrix $\mathcal{Q} = [h_{ij}]$ as

$$h_{ij} = \mathbf{i}^{-\frac{k(k-1)}{2} - k(i+j+r-1)} q^{\frac{k(k-1)}{4} + \frac{k(i+j+r-1)}{2}} \frac{(1-q)^k \,(q;q)_{i+j+r-1}}{(q;q)_{i+j+k+r-1}}.$$

We call the matrix H_n the q-Pilbert (=Pochhammer-H*ilbert*) matrix.

Furthermore, we will use *Fibonomial coefficients*

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{F_n F_{n-1} \dots F_{n-k+1}}{F_1 \dots F_k}.$$

The link between the two notations is

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix} \quad \text{with} \quad q = -\alpha^{-2}.$$

In the sequel, we list all our results. Proofs are given in the following section, and they are all applications of the q-version of Zeilberger's algorithm. This link between mathematics and computer proofs makes this article an appropriate choice for the present journal.

We will obtain the LU-decomposition $\mathcal{Q} = L \cdot U$:

Theorem 1. *For $1 \le d \le n$ we have*

$$L_{n,d} = \mathbf{i}^{k(d-n)} q^{\frac{k(n-d)}{2}} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix} \begin{bmatrix} 2d+k+r-1 \\ d+r \end{bmatrix} \begin{bmatrix} n+d+r+k-1 \\ n+r \end{bmatrix}^{-1}$$

and its Fibonacci corollary

$$L_{n,d} = \left\{ \begin{matrix} n-1 \\ d-1 \end{matrix} \right\} \left\{ \begin{matrix} 2d+k+r-1 \\ d+r \end{matrix} \right\} \left\{ \begin{matrix} n+d+k+r-1 \\ n+r \end{matrix} \right\}^{-1}.$$

Theorem 2. *For $1 \le d \le n$ we have*

$$U_{d,n} = \mathbf{i}^{\frac{k}{2}(3-k) - k(n+r+d)} q^{\frac{k}{2}\left(d+n+r-\frac{3}{2}+\frac{k}{2}\right) - r - d + dr + d^2} \frac{(1-q)^k}{(1-q^n)\,(q;q)_{k-1}}$$
$$\times \begin{bmatrix} 2d+r+k-2 \\ d+r \end{bmatrix}^{-1} \begin{bmatrix} n+d+r+k-1 \\ n \end{bmatrix}^{-1} \begin{bmatrix} n+r \\ d+r \end{bmatrix}$$

and its Fibonacci corollary

$$U_{d,n} = (-1)^{r(d+1)} \left\{ \begin{matrix} 2d+r+k-2 \\ d+r \end{matrix} \right\}^{-1} \left\{ \begin{matrix} n+d+r+k-1 \\ n \end{matrix} \right\}^{-1} \left\{ \begin{matrix} n+r \\ d+r \end{matrix} \right\}$$
$$\times \frac{1}{F_n} \left(\prod_{i=1}^{k-1} F_i \right)^{-1}.$$

We could also determine the inverses of the matrices L and U:

Theorem 3. *For $1 \le d \le n$ we have*

$$L_{n,d}^{-1} = \mathbf{i}^{(k+2)(d-n)} q^{\frac{1}{2}(d-n)(d-k-n+1)} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix} \begin{bmatrix} n+d+r+k-2 \\ d+r \end{bmatrix} \times \begin{bmatrix} 2n+r+k-2 \\ n+r \end{bmatrix}^{-1}$$

and its Fibonacci corollary

$$L_{n,d}^{-1} = (-1)^{(n+1)d+\frac{n(n+1)}{2}+\frac{d(d+1)}{2}} \left\{ \begin{matrix} n-1 \\ d-1 \end{matrix} \right\} \left\{ \begin{matrix} n+d+r+k-2 \\ d+r \end{matrix} \right\} \times \left\{ \begin{matrix} 2n+r+k-2 \\ n+r \end{matrix} \right\}^{-1}.$$

Theorem 4. *For $1 \le d \le n$ we have*

$$U_{d,n}^{-1} = (-1)^{\frac{k(d+n+r)}{2}-d+\frac{k(k-3)}{4}+n^2} q^{-\frac{n(n-1)}{2}+r-\frac{k(d+n+r)}{2}-n(d+r)+\frac{d(d+1)}{2}-\frac{k(k-3)}{4}} \times \begin{bmatrix} 2n+r+k-1 \\ n \end{bmatrix} \begin{bmatrix} n+d+r+k-2 \\ d+r \end{bmatrix} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix} \frac{(1-q^n)\,(q;q)_{k-1}}{(1-q)^k}$$

and its Fibonacci corollary

$$U_{d,n}^{-1} = (-1)^{\frac{n(n+1)}{2}+\frac{d(d-1)}{2}-n(d+r)+r} F_n \left(\prod_{v=1}^{k-1} F_v \right) \times \left\{ \begin{matrix} 2n+r+k-1 \\ n \end{matrix} \right\} \left\{ \begin{matrix} n+d+r+k-2 \\ d+r \end{matrix} \right\} \left\{ \begin{matrix} n-1 \\ d-1 \end{matrix} \right\}.$$

As a consequence, we can compute the determinant of $\mathcal{Q}_n$, since it is simply evaluated as $U_{1,1} \cdots U_{n,n}$ (we only state the Fibonacci version):

Theorem 5.

$$\det \mathcal{Q}_n = \frac{(-1)^{\frac{1}{2}nr(n+3)}}{\left(\prod_{v=1}^{k-1} F_v \right)^n} \prod_{d=1}^{n} \left\{ \begin{matrix} 2d+k+r-2 \\ d+r \end{matrix} \right\}^{-1} \left\{ \begin{matrix} 2d+k+r-1 \\ d \end{matrix} \right\}^{-1} \frac{1}{F_d}.$$

Many similar (and most of them much deeper) determinant evaluations can be found in [2]; not surprisingly, LU-decomposition is one of the important tools in this impressive survey paper.

Now we compute the inverse of the matrix $\mathcal{Q}$. This time it depends on the dimension, so we compute $(\mathcal{Q}_n)^{-1}$.

Theorem 6. *For $1 \le i, j \le n$:*

$$(\mathcal{Q}_n)_{i,j}^{-1} = \frac{(q;q)_{k-1} q^{\frac{i^2}{2}-\frac{(k-1)i}{2}-\frac{kr}{2}+r-\frac{k^2}{4}+\frac{3k}{4}} (-1)^{i+j} \mathbf{i}^{kr+ik+\frac{k^2}{2}-\frac{3k}{2}+\frac{j^2}{2}+\frac{j(k+1)}{2}}}{(q;q)_{i+r}(q;q)_{i-1}(q;q)_{j-1}(q;q)_{j+r}(1-q)^k} \times \sum_{\max\{i,j\} \le h \le n} \frac{(1-q^{2h+r+k-1})(q;q)_{h-1}(q;q)_{i+h+r+k-2}(q;q)_{h+r}}{(q;q)_{h-i}(q;q)_{h-j}(q;q)_{h+r+k-1}(q;q)_{h+k-2}} q^{-hi-hj-hr}.$$

Finally, we provide the Cholesky decomposition.

Theorem 7. *For $i, j \geq 1$:*

$$\mathcal{C}_{i,j} = \frac{(q;q)_{i+r}(q;q)_{i-1}}{(q;q)_{i+j+k+r-1}(q;q)_{i-j}}(1-q)^{\frac{k}{2}} q^{\frac{j(j-1)}{2}+\frac{ki}{2}+\frac{k(k-3)}{8}+\frac{rj}{2}+\frac{kr}{4}-\frac{r}{2}} \times \mathbf{i}^{-\frac{k(k-3)}{4}+\frac{3rk}{2}-ik} \sqrt{\frac{(1-q^{2j+k+r-1})(q;q)_{j+k-2}(q;q)_{j+k+r-1}}{(q;q)_{k-1}(q;q)_{j-1}(q;q)_{j+r}}}.$$

2. Proofs

In order to show that indeed $\mathcal{Q} = L \cdot U$, we need to show that for any m, n:

$$\sum_d L_{m,d} U_{d,n} = \mathcal{Q}_{m,n} = \alpha^{-(m+n+r-1)k-\frac{k(k-1)}{2}} \frac{(q;q)_{m+n+r+k-1}}{(q;q)_{m+n+r-1}(1-q)^k}.$$

In rewritten form the formula to be proved reads

$$\sum_d q^{(d-1)(d+r)} \begin{bmatrix} m-1 \\ d-1 \end{bmatrix} \begin{bmatrix} 2d+k+r-1 \\ d+r \end{bmatrix} \begin{bmatrix} m+d+r+k-1 \\ m+r \end{bmatrix}^{-1} \times \begin{bmatrix} 2d+r+k-2 \\ d+r \end{bmatrix}^{-1} \begin{bmatrix} n+d+k+r-1 \\ n \end{bmatrix}^{-1} \begin{bmatrix} n+r \\ d+r \end{bmatrix} = \begin{bmatrix} m+n+k+r-1 \\ k \end{bmatrix}^{-1} \frac{1-q^n}{1-q^k}. \tag{1}$$

For the verification of the last equation, let us denote the LHS of the equation (1) by $\mathtt{SUM}_m$, then the Mathematica version of the q-Zeilberger algorithm [3] produces the recursion

$$\mathtt{SUM}_m = \frac{1-q^{m+n+r-1}}{1-q^{k+m+n+r-1}} \mathtt{SUM}_{m-1}.$$

(For $m \neq 1$, $k+m+n+r-1 \neq 0$.) So we compute (directly from the definition)

$$\mathtt{SUM}_1 = q^{(1+r)} \begin{bmatrix} r+k \\ 1+r \end{bmatrix}^{-1} \begin{bmatrix} n+k+r \\ n \end{bmatrix}^{-1} \begin{bmatrix} n+r \\ 1+r \end{bmatrix} = \frac{(1-q^n)(q;q)_{k-1}(q;q)_{n+r}}{(q;q)_{n+k+r}}$$

and get

$$\begin{aligned} \mathtt{SUM}_m &= \frac{(q;q)_{m+n+r-1}(q;q)_{k+n+r}}{(q;q)_{n+r}(q;q)_{k+m+n+r-1}} \frac{(1-q^n)(q;q)_{k-1}(q;q)_{n+r}}{(q;q)_{n+k+r}} \\ &= \frac{(q;q)_{m+n+r-1}}{(q;q)_{k+m+n+r-1}}(1-q^n)(q;q)_{k-1} \\ &= \begin{bmatrix} m+n+k+r-1 \\ k \end{bmatrix}^{-1} \frac{1-q^n}{1-q^k}. \blacksquare \end{aligned}$$

Note that nowadays, such identities are a routine verification using the q-Zeilberger algorithm, as described in the book [4].

For interest, we also state (as a corollary) the corresponding Fibonacci identity:

$$\sum_d (-1)^{r(d+1)} \begin{Bmatrix} m-1 \\ d-1 \end{Bmatrix} \begin{Bmatrix} 2d+k+r-1 \\ d+r \end{Bmatrix} \begin{Bmatrix} m+d+r+k-1 \\ m+r \end{Bmatrix}^{-1}$$
$$\times \begin{Bmatrix} 2d+r+k-2 \\ d+r \end{Bmatrix}^{-1} \begin{Bmatrix} n+d+k+r-1 \\ n \end{Bmatrix}^{-1} \begin{Bmatrix} n+r \\ d+r \end{Bmatrix}$$
$$= \frac{F_n}{F_k} \begin{Bmatrix} m+n+k+r-1 \\ k \end{Bmatrix}^{-1}.$$

Now we move to the inverse matrices. Since L and L^{-1} are lower triangular matrices, we only need to look at the entries indexed by (m,n) with $m \geq n$:

$$\sum_{n\leq d\leq m} L_{m,d} L^{-1}_{d,n} = \sum_{n\leq d\leq m} (-1)^{d+n}\, \mathbf{i}^{k(3m+n)} q^{\frac{(n-d)^2+k(m-n)+(n-d)}{2}}$$
$$\times \begin{bmatrix} m-1 \\ d-1 \end{bmatrix} \begin{bmatrix} 2d+k+r-1 \\ d+r \end{bmatrix} \begin{bmatrix} m+d+r+k-1 \\ m+r \end{bmatrix}^{-1}$$
$$\times \begin{bmatrix} d-1 \\ n-1 \end{bmatrix} \begin{bmatrix} n+d+r+k-2 \\ n+r \end{bmatrix} \begin{bmatrix} 2d+r+k-2 \\ d+r \end{bmatrix}^{-1}$$
$$= (-1)^n\, \mathbf{i}^{k(n-m)} \frac{(q;q)_{m-1}}{(q;q)_{n-1}} \frac{(q;q)_{m+r}}{(q;q)_{n+r}}$$
$$\times \sum_{n\leq d\leq m} (-1)^d q^{\frac{(n-d)^2+k(m-n)+(n-d)}{2}}$$
$$\times \frac{(1-q^{2d+k+r-1})\,(q;q)_{d+k+n+r-2}}{(q;q)_{m-d}\,(q;q)_{d-n}\,(q;q)_{d+k+m+r-1}}.$$

For the sum in the last expression, that is,

$$\sum_{n\leq d\leq m} (-1)^d q^{\frac{(n-d)^2+k(m-n)+(n-d)}{2}} \frac{(1-q^{2d+k+r-1})\,(q;q)_{d+k+n+r-2}}{(q;q)_{m-d}\,(q;q)_{d-n}\,(q;q)_{d+k+m+r-1}},$$

the q-Zeilberger algorithm evaluates it and give us 0 for $m \neq n$. For $m = n$, it is easy:

$$(-1)^n(1-q^{2n+k+r-1}) \frac{(q;q)_{2n+k+r-2}}{(q;q)_{2n+k+r-1}} = (-1)^n.$$

In that case, the equality is valid as well and so the proof is complete. ∎

Its Fibonacci corollary is

$$(-1)^{\frac{(n+1)n}{2}} \sum_{n\leq d\leq m} (-1)^{\frac{(d+1)d}{2}} \frac{F_{2d+r+k-1}}{F_{d+k-1}} \begin{Bmatrix} m-1 \\ d-1 \end{Bmatrix} \begin{Bmatrix} d-1 \\ n-1 \end{Bmatrix}$$
$$\times \begin{Bmatrix} n+d+r+k-2 \\ n+r \end{Bmatrix} \begin{Bmatrix} m+d+k+r-1 \\ m+r \end{Bmatrix}^{-1} = \delta_{n,m}$$

where $\delta_{n,m}$ stands for the Kronecker delta.

A similar argument for $U\cdot U^{-1}$ is as follows:

$$\begin{aligned}\sum_{m\le d\le n} U_{m,d}U^{-1}_{d,n} &= (-1)^n\,\mathbf{i}^{k(n-m)}q^{-m+\frac12 n+\frac12 km-\frac12 kn+mr-nr+m^2-\frac12 n^2}\,(1-q^n)\\ &\quad\times \begin{bmatrix}2n+r+k-1\\ n\end{bmatrix}\begin{bmatrix}2m+r+k-2\\ m+r\end{bmatrix}^{-1}\\ &\quad\times\sum_{m\le d\le n}(-1)^d\,q^{\frac12 d(d+1)-dn}\frac{1}{1-q^d}\begin{bmatrix}d+m+r+k-1\\ d\end{bmatrix}^{-1}\\ &\quad\times\begin{bmatrix}n-1\\ d-1\end{bmatrix}\begin{bmatrix}d+r\\ m+r\end{bmatrix}\begin{bmatrix}n+d+r+k-2\\ d+r\end{bmatrix}.\end{aligned}$$

We evaluate the sum in the last equation

$$\begin{aligned}\sum_{m\le d\le n}(-1)^d\,q^{\frac12 d(d+1)-dn}\frac{1}{1-q^d}\begin{bmatrix}d+m+r+k-1\\ d\end{bmatrix}^{-1}\begin{bmatrix}n-1\\ d-1\end{bmatrix}&\\ \times\begin{bmatrix}d+r\\ m+r\end{bmatrix}\begin{bmatrix}n+d+r+k-2\\ d+r\end{bmatrix}&,\end{aligned}$$

using the Mathematica version of the q-Zeilberger algorithm. We get that the sum is $=0$ provided that $m\ne n$ and $k+m+n+r\ne 1$. If $m=n$, it is easy to evaluate:

$$(-1)^n\,q^{\frac12 n(n+1)-n^2}\frac{1}{1-q^n}\begin{bmatrix}2n+r+k-1\\ n\end{bmatrix}^{-1}\begin{bmatrix}n-1\\ n-1\end{bmatrix}\begin{bmatrix}n+r\\ n+r\end{bmatrix}\begin{bmatrix}2n+r+k-2\\ n+r\end{bmatrix},$$

or simpler

$$(-1)^n\,q^{-\frac12 n(n-1)}\frac{(q;q)_{n-1}(q;q)_{n+r+k-1}}{(1-q^{2n+r+k-1})(q;q)_{n+r}(q;q)_{n+k-2}}.\quad\blacksquare$$

Now we turn to the inverse matrix. Since

$$L^{-1}_{ij}=\frac{(q;q)_{i+j+r+k-2}(q;q)_{i-1}(q;q)_{i+r}}{(q;q)_{2i+r+k-2}(q;q)_{j-1}(q;q)_{j+r}(q;q)_{i-j}}\frac{q^{\frac{i^2}{2}+\frac{j^2}{2}+\frac{(i-j)(k-1)}{2}-ij}}{\mathbf{i}^{(i-j)(k-2)}}$$

and

$$\begin{aligned}U^{-1}_{ij}&=\frac{(q;q)_{2j+r+k-1}(q;q)_{i+j+r+k-2}(q;q)_{k-1}}{(q;q)_{j-i}(q;q)_{j+r+k-1}(q;q)_{i+r}(q;q)_{j+k-2}(q;q)_{i-1}(1-q)^k}\\ &\quad\times q^{\frac{i^2}{2}-\frac{j^2}{2}-ij-jr-\frac{(k-1)j}{2}-\frac{(k-1)i}{2}-\frac{kr}{2}+r-\frac{k^2}{4}+\frac{3k}{4}}\,\mathbf{i}^{kr+(i+j)(k+2)+\frac{k^2}{2}-\frac{3k}{2}},\end{aligned}$$

we write

$$\begin{aligned}\left((\mathcal{Q}_n)^{-1}\right)_{i,j}&=\sum_h U^{-1}_{ih}L^{-1}_{hj}\\ &=\sum_h\frac{(q;q)_{2h+r+k-1}(q;q)_{i+h+r+k-2}(q;q)_{k-1}}{(q;q)_{h-i}(q;q)_{h+r+k-1}(q;q)_{i+r}(q;q)_{h+k-2}(q;q)_{i-1}(1-q)^k}\\ &\quad\times q^{\frac{i^2}{2}+\frac{j^2}{2}-h(i+j+r)-\frac{(k-1)i}{2}-\frac{kr}{2}+r-\frac{k^2}{4}+\frac{3k}{4}+\frac{j}{2}-\frac{jk}{2}}\,(-1)^{i-j}\,\mathbf{i}^{k(i+j+r)+\frac{k(k-3)}{2}}\\ &\quad\times\frac{(q;q)_{h+j+r+k-2}(q;q)_{h-1}(q;q)_{h+r}}{(q;q)_{2h+r+k-2}(q;q)_{j-1}(q;q)_{j+r}(q;q)_{h-j}}.\end{aligned}$$

The final formula as given in the theorem follows from some straightforward simplifications. Unfortunately, the sum cannot be evaluated in closed form.

For the proof of the Cholesky decomposition, we need this formula:

$$\sum_{1\le j\le\min\{i,l\}} \mathcal{C}_{i,j}\mathcal{C}_{l,j} = \frac{(q;q)_{i+l+r-1}(1-q)^k q^{\frac{(i+l+r)k}{2}+\frac{k(k-3)}{4}}}{(q;q)_{i+l+k+r-1}\mathbf{i}^{(i+l+r)k+\frac{k(k-3)}{2}}}.$$

After some straightforward simplifications, it means that we must show that

$$\begin{aligned}&\sum_{1\le j\le\min\{i,l\}} q^{(j+r)(j-1)}\\ &\times \frac{(1-q^{2j+k+r-1})(q;q)_{j+k-2}(q;q)_{j+k+r-1}}{(q;q)_{j-1}(q;q)_{j+r}(q;q)_{i+j+k+r-1}(q;q)_{i-j}(q;q)_{l+j+k+r-1}(q;q)_{l-j}}\\ &= \frac{(q;q)_{i+l+r-1}(q;q)_{k-1}}{(q;q)_{i+l+k+r-1}(q;q)_{i+r}(q;q)_{i-1}(q;q)_{l+r}(q;q)_{l-1}}.\end{aligned}$$

Denoting the LHS by $\mathtt{SUM}_i$, the q-Zeilberger algorithm gives the answer

$$\mathtt{SUM}_i = \frac{(1-q^{i+l+r-1})}{(1-q^{i-1})(1-q^{i+r})(1-q^{i+k+l+r-1})}\mathtt{SUM}_{i-1}$$

for $i \ne 1$, $i+k+l+r \ne 1$.

Now we compute $\mathtt{SUM}_1$:

$$\mathtt{SUM}_1 = \frac{(q;q)_{k-1}}{(q;q)_{r+1}(q;q)_{l+k+r}(q;q)_{l-1}}.$$

Therefore

$$\begin{aligned}\mathtt{SUM}_i &= \frac{(q;q)_{i+l+r-1}(q;q)_{r+1}(q;q)_{k+l+r}}{(q;q)_{l+r}(q;q)_{i-1}(q;q)_{i+r}(q;q)_{i+k+l+r-1}}\frac{(q;q)_{k-1}}{(q;q)_{r+1}(q;q)_{l+k+r}(q;q)_{l-1}}\\ &= \frac{(q;q)_{i+l+r-1}(q;q)_{k-1}}{(q;q)_{l+r}(q;q)_{i-1}(q;q)_{i+r}(q;q)_{i+k+l+r-1}(q;q)_{l-1}},\end{aligned}$$

as claimed.

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