ON q–OLIVIER FUNCTIONS

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Abstract. We consider words $w_1 \dots w_n$ with letters $w_i \in \{1, 2, 3, \dots\}$ satisfying an up–up–down pattern like $a_1 \le a_2 \le a_3 \ge a_4 \le a_5 \le a_6 \ge \dots$. Attaching the (geometric) probability pq^{i-1} to the letter i (with $p = 1-q$), every word gets a probability by assuming indepence of letters. We are interested in the probability that a random word of length n satisfies the up–up–down condition. It turns out that one has to consider the 3 residue classes $\pmod 3$ separately; then one can compute the associated probability generating function. They turn out to be q–analogues of so called Olivier functions.

1. Introduction

We consider words $w_1 \dots w_n$ with letters $w_i \in \{1, 2, 3, \dots\}$ satisfying an up–up–down pattern like $a_1 \le a_2 \le a_3 \ge a_4 \le a_5 \le a_6 \ge \dots$. Attaching the (geometric) probability pq^{i-1} to the letter i (with $p = 1 - q$), every word gets a probability by assuming independence of letters. We are interested in the probability that a random word of length n satisfies the up–up–down condition. It turns out that one has to consider the 3 residue classes $\pmod 3$ separately; then one can compute the associated probability generating function.

This extends previous research of the first author about up-down patterns, leading to q–tangent and q–secant numbers, see [4]. There are several variations possible, since "up" might mean $\le$ or $<$ and similar for "down." Also, one can consider down–down–up patterns.

As it was discussed already in [4], the limit $q \to 1$ yields the model of permutations (in one line notation). Thus we get the generating functions of permutations satisfying the up–up–down condition as corollaries. These results are due to Carlitz [1]. The generating functions he obtained are known as Olivier functions [3, 2]:

$$\Phi_{k,t}(z) := \sum_{n \ge 0} \frac{z^{kn+t}}{(kn+t)!}.$$

In this way we get various different q–Olivier functions.

We arrange these functions in a table and prove only one significant instance, the other ones being similar. More details can be found in the forthcoming thesis of the second author [7].

Extensions to patterns like $(\le \cdots \le \ge)^*$ and $(\le\le>>)^*$ are also discussed.

We need the following notations: $(x; q)_n := (1 - x)(1 - qx) \dots (1 - q^{n-1}x)$ and $[n]_q! = (q; q)_n / p^n$, which in the limit $q \to 1$ are the ordinary factorials.

2. Results

We first consider $n \equiv 1 \bmod 3$. The probability generating function is given by

$$\sum_{n \ge 0} (-1)^n \frac{z^{3n+1} q^{An^2+Bn}}{[3n+1]_q!} \Big/ \sum_{n \ge 0} (-1)^n \frac{z^{3n} q^{Cn^2+Dn}}{[3n]_q!},$$

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with the parameters A, B, C, D given in the following table.

	A	B	C	D
$(\le\le\ge)^*$	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$
$(\le\le>)^*$	0	0	0	0
$(\le<\ge)^*$	3	0	3	-2
$(\le<>)^*$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$
$(<\le\ge)^*$	3	1	3	-1
$(<\le>)^*$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
$(<<\ge)^*$	$\frac{9}{2}$	$\frac{3}{2}$	$\frac{9}{2}$	$-\frac{3}{2}$
$(<<>)^*$	3	2	3	0

	A	B	C	D
$(\ge\ge\le)^*$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$
$(\ge\ge<)^*$	0	0	0	0
$(\ge>\le)^*$	3	2	3	-1
$(\ge><)^*$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
$(>\ge\le)^*$	3	1	3	-2
$(>\ge<)^*$	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$
$(>>\le)^*$	$\frac{9}{2}$	$\frac{3}{2}$	$\frac{9}{2}$	$-\frac{3}{2}$
$(>><)^*$	3	0	3	0

Now we consider $n \equiv 2 \bmod 3$. The probability generating function is given by

$$\sum_{n\ge0}(-1)^n\frac{z^{3n+2}q^{An^2+Bn+E}}{[3n+2]_q!}\Bigg/\sum_{n\ge0}(-1)^n\frac{z^{3n}q^{Cn^2+Dn}}{[3n]_q!},$$

with the parameters A, B, C, D, E given in the following table.

	A	B	C	D	E
$(\le\le\ge)^*\le$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	0
$(\le\le>)^*\le$	0	0	0	0	0
$(\le<\ge)^*\le$	3	2	3	-2	0
$(\le<>)^*\le$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	0
$(<\le\ge)^*<$	3	3	3	-1	1
$(<\le>)^*<$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	1
$(<<\ge)^*<$	$\frac{9}{2}$	$\frac{9}{2}$	$\frac{9}{2}$	$-\frac{3}{2}$	1
$(<<>)^*<$	3	4	3	0	1

	A	B	C	D	E
$(\ge\ge\le)^*\ge$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	0
$(\ge\ge<)^*\ge$	0	0	0	0	0
$(\ge>\le)^*\ge$	3	2	3	-1	0
$(\ge><)^*\ge$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	0
$(>\ge\le)^*>$	3	4	3	-2	1
$(>\ge<)^*>$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	1
$(>>\le)^*>$	$\frac{9}{2}$	$\frac{9}{2}$	$\frac{9}{2}$	$-\frac{3}{2}$	1
$(>><)^*>$	3	3	3	0	1

Finally, we consider $n \equiv 0 \bmod 3$. The probability generating function is given by

$$1\Bigg/\sum_{n\ge0}(-1)^n\frac{z^{3n}q^{Cn^2+Dn}}{[3n]_q!},$$

with the parameters C, D given in the following table.

	C	D
$(\le\le\ge)^*\le\le$	$\frac{3}{2}$	$-\frac{3}{2}$
$(\le\le>)^*\le\le$	0	0
$(\le<\ge)^*\le<$	3	-2
$(\le<>)^*\le<$	$\frac{3}{2}$	$-\frac{1}{2}$
$(<\le\ge)^*<\le$	3	-1
$(<\le>)^*<\le$	$\frac{3}{2}$	$\frac{1}{2}$
$(<<\ge)^*<<$	$\frac{9}{2}$	$-\frac{3}{2}$
$(<<>)^*<<$	3	0

	C	D
$(\ge\ge\le)^*\ge\ge$	$\frac{3}{2}$	$-\frac{3}{2}$
$(\ge\ge<)^*\ge\ge$	0	0
$(\ge>\le)^*\ge>$	3	-1
$(\ge><)^*\ge>$	$\frac{3}{2}$	$\frac{1}{2}$
$(>\ge\le)^*>\ge$	3	-2
$(>\ge<)^*>\ge$	$\frac{3}{2}$	$-\frac{1}{2}$
$(>>\ge)^*>>$	$\frac{9}{2}$	$-\frac{3}{2}$
$(>><)^*>>$	3	0

As an illustration, we prove the instance of $(\le\le\ge)^*$, i. e. we consider words of length $1,4,7,\dots$. Denote by $a_n(u)$ the following probability generating function: the coefficient of u^i in $a_n(u)$ is the probability that a word of length $3n-2$ has the desired shape and ends with i. The initial value is immediate:

$$a_1(u)=\frac{pu}{1-qu}.$$

To get a recursion, we compute what can become of the variable u^i:

$$\begin{aligned} u^i &\longrightarrow \sum_{i\le j\le k\ge l} pq^{j-1}pq^{k-1}pq^{l-1}u^l \\ &=\frac{p^3uq^{2i}}{q^2(1-q)(1-q^2)(1-qu)}-\frac{up^3u^iq^{3i}}{q^2(1-qu)(1-q^2u)(1-q^3u)}. \end{aligned} \tag{1}$$

The multiple sum was computed with Maple. This gives us the following recursion which involves substitutions as well:

$$a_{n+1}(u)=\frac{p^3u}{q^2(q;q)_2(1-qu)}a_n(q^2)-\frac{p^3u}{q^2(qu;q)_3}a_n(q^3u).$$

We translate that into a generating function: Set

$$f(u)=\sum_{n\ge1}a_n(u)z^{3n-2},$$

then

$$f(u)=z\frac{pu}{1-qu}+\frac{z^3p^3u}{q^2(q;q)_2(1-qu)}f(q^2)-\frac{z^3p^3u}{q^2(qu;q)_3}f(q^3u).$$

The nature of that functional equation makes it necessary to compute first $f(q^2)$, something that can be achieved by iteration:

$$\begin{aligned} f(q^2)&=z\frac{pq^2}{1-q^3}+\frac{z^3p^3q^2}{q^2(q;q)_2(1-q^3)}f(q^2)\\ &\quad-\frac{z^3p^3q^2}{q^2(q^3;q)_3}\Bigg[z\frac{pq^5}{1-q^6}+\frac{z^3p^3q^5}{q^2(q;q)_2(1-q^6)}f(q^2)\\ &\quad-\frac{z^3p^3q^5}{q^2(q^6;q)_3}\Bigg[z\frac{pq^8}{1-q^9}+\frac{z^3p^3q^8}{q^2(q;q)_2(1-q^9)}f(q^2)\\ &\quad-\frac{z^3p^3q^8}{q^2(q^9;q)_3}\Bigg[\dots, \end{aligned}$$

or

$$f(q^2)\Big[1-\frac{z^3p^3}{(q;q)_3}+\frac{z^6p^6q^3}{(q;q)_6}-\frac{z^9p^9q^{0+3+6}}{(q;q)_9}+\dots\Big]$$
$$=\frac{zpq^2}{(q^3;q)_1}-\frac{z^4p^4q^5}{(q^3;q)_4}+\frac{z^7p^7q^{11}}{(q^3;q)_7}-\frac{z^{10}p^{10}q^{2+(0+3+6+9)}}{(q^3;q)_{10}}+\dots,$$

which gives us the explicit form

$$f(q^2)=\sum_{n\ge0}(-1)^n\frac{z^{3n+1}p^{3n+1}q^{\frac32n^2+\frac32n+2}}{(q^3;q)_{3n+1}}\Big/\sum_{n\ge0}(-1)^n\frac{z^{3n}p^{3n}q^{\frac32n^2-\frac32n}}{(q;q)_{3n}}.$$

Now that this quantity is known, we can compute $f(u)$, again by iteration:

$$\begin{aligned}f(u)&=z\frac{pu}{1-qu}+\frac{z^3p^3u}{q^2(q;q)_2(1-qu)}f(q^2)\\&-\frac{z^3p^3u}{q^2(qu;q)_3}\Big[z\frac{puq^3}{1-q^4u}+\frac{z^3p^3q^3u}{q^2(q;q)_2(1-q^4u)}f(q^2)\\&-\frac{z^3p^3q^3u}{q^2(q^4u;q)_3}\Big[z\frac{puq^6}{1-q^7u}+\frac{z^3p^3q^6u}{q^2(q;q)_2(1-q^7u)}f(q^2)\\&-\frac{z^3p^3q^6u}{q^2(q^7;q)_3}\Big[\dots,\end{aligned}$$

or

$$f(u)=\sum_{n\ge0}(-1)^n\frac{z^{3n+1}p^{3n+1}u^{n+1}q^{\frac32n^2-\frac12n}}{(qu;q)_{3n+1}}-f(q^2)\sum_{n\ge1}(-1)^n\frac{z^{3n}p^{3n}u^nq^{\frac32n^2-\frac72n}}{(q;q)_2(qu;q)_{3n-2}}.$$

Of course, we are especially interested in $u=1$:

$$\begin{aligned}f(1)&=\sum_{n\ge0}(-1)^n\frac{z^{3n+1}p^{3n+1}q^{\frac32n^2-\frac12n}}{(q;q)_{3n+1}}\\&-\sum_{n\ge0}(-1)^n\frac{z^{3n+1}p^{3n+1}q^{\frac32n^2+\frac32n+2}}{(q;q)_{3n+3}}\sum_{n\ge1}(-1)^n\frac{z^{3n}p^{3n}q^{\frac32n^2-\frac72n}}{(q;q)_{3n-2}}\Big/\sum_{n\ge0}(-1)^n\frac{z^{3n}p^{3n}q^{\frac32n^2-\frac32n}}{(q;q)_{3n}}.\end{aligned}$$

We will write everything over a common denominator and thus have to compute

$$\begin{aligned}&\sum_{n\ge0}(-1)^n\frac{z^{3n+1}p^{3n+1}q^{\frac32n^2-\frac12n}}{(q;q)_{3n+1}}\sum_{n\ge0}(-1)^n\frac{z^{3n}p^{3n}q^{\frac32n^2-\frac32n}}{(q;q)_{3n}}\\&\qquad+\sum_{n\ge0}(-1)^{n+1}\frac{z^{3n+1}p^{3n+1}q^{\frac32n^2+\frac32n+2}}{(q;q)_{3n+3}}\sum_{n\ge1}(-1)^n\frac{z^{3n}p^{3n}q^{\frac32n^2-\frac72n}}{(q;q)_{3n-2}}.\end{aligned}$$

The coefficient of $(pz)^{3n+1}$ in the first sum is

$$(-1)^n\sum_{k=0}^n\frac{q^{\frac32k^2-\frac12k}}{(q;q)_{3k+1}}\frac{q^{\frac32(n-k)^2-\frac32(n-k)}}{(q;q)_{3n-3k}}=(-1)^nq^{\frac32n(n-1)}\sum_{k=0}^n\frac{q^{k(3k+1)-3kn}}{(q;q)_{3k+1}(q;q)_{3n-3k}};$$

the coefficient of $(pz)^{3n+1}$ in the second sum is

$$(-1)^{n+1}q^{\frac32n(n-1)}\sum_{k=0}^n\frac{q^{3k^2-k(3n-1)}}{(q;q)_{3k+1}(q;q)_{3n-3k}}+(-1)^n\frac{q^{\frac32n^2-\frac12n}}{(q;q)_{3n+1}};$$

so the combination of both is simply

$$(-1)^n \frac{q^{\frac{3}{2}n^2-\frac{1}{2}n}}{(q;q)_{3n+1}}.$$

This proves that

$$f(1) = \sum_{n\ge0}(-1)^n \frac{z^{3n+1}p^{3n+1}q^{\frac{3}{2}n^2-\frac{1}{2}n}}{(q;q)_{3n+1}} \bigg/ \sum_{n\ge0}(-1)^n \frac{z^{3n}p^{3n}q^{\frac{3}{2}n^2-\frac{3}{2}n}}{(q;q)_{3n}},$$

as claimed.

3. A generalization

We briefly report on the more general pattern up... up–down, with $k-1$ up steps always followed by a down step (except possibly at the right border). Details of these computations are to be found in [7]. There are four cases to be considered:

First, we consider $n \equiv 0 \bmod k$. For the pattern $(\le\le\cdots\le\ge)^*$, the probability generating function is

$$1 \bigg/ \sum_{n\ge0}(-1)^n \frac{z^{kn}q^{\frac{k}{2}n^2-\frac{k}{2}n}}{[kn]_q!}.$$

For the pattern $(\le\le\cdots\le>)^*$, the probability generating function is

$$1 \bigg/ \sum_{n\ge0}(-1)^n \frac{z^{kn}}{[kn]_q!}.$$

For the pattern $(<<\cdots<\ge)^*$, the probability generating function is

$$1 \bigg/ \sum_{n\ge0}(-1)^n \frac{z^{kn}q^{\frac{k^2}{2}n^2-\frac{k}{2}n}}{[kn]_q!}.$$

Finally, for the pattern $(<<\cdots<>)^*$, the probability generating function is given by

$$1 \bigg/ \sum_{n\ge0}(-1)^n \frac{z^{kn}q^{\frac{k(k-1)}{2}n^2}}{[kn]_q!}.$$

Next, we consider $n \equiv t \bmod k$, where $t = 1, 2, \dots, k-1$. That means that we consider $t-1$ extra up steps at the right border. For the pattern $(\le\le\cdots\le\ge)^* \le^{t-1}$, the probability generating function is

$$\sum_{n\ge0}(-1)^n \frac{z^{kn+t}q^{\frac{k}{2}n^2+\left(t-\frac{k}{2}\right)n}}{[kn+t]_q!} \bigg/ \sum_{n\ge0}(-1)^n \frac{z^{kn}q^{\frac{k}{2}n^2-\frac{k}{2}n}}{[kn]_q!}.$$

For the pattern $(\le\le\cdots\le>)^* \le^{t-1}$, the probability generating function is

$$\sum_{n\ge0}(-1)^n \frac{z^{kn+t}}{[nk+t]_q!} \bigg/ \sum_{n\ge0}(-1)^n \frac{z^{kn}}{[kn]_q!}.$$

For the pattern $(<<\cdots<\ge)^* <^{t-1}$, the probability generating function is

$$\sum_{n\ge0}(-1)^n \frac{z^{kn+t}q^{\frac{k^2}{2}n^2+(t-\frac{1}{2})kn+\frac{t(t-1)}{2}}}{[kn+t]_q!} \bigg/ \sum_{n\ge0}(-1)^n \frac{z^{kn}q^{\frac{k^2}{2}n^2-\frac{k}{2}n}}{[kn]_q!}.$$

Finally, for the pattern $(<<\cdots<>)^*<^{t-1}$, the probability generating function is given by

$$\sum_{n\ge0}(-1)^n\frac{z^{kn+t}q^{\frac{k(k-1)}{2}n^2+(k-1)tn+\frac{t(t-1)}{2}}}{[nk+t]_q!}\Bigg/\sum_{n\ge0}(-1)^n\frac{z^{kn}q^{\frac{k(k-1)}{2}n^2}}{[kn]_q!}.$$

We sketch again the instance $(\le\le\cdots\le\ge)^*$: Maple computes in analogy to (1)

$$\begin{aligned}u^i\longrightarrow&\sum_{i\le j\le k\le l\ge h}pq^{j-1}pq^{k-1}pq^{l-1}pq^{h-1}u^h\\&=\frac{p^4uq^{3i}}{q^3(q;q)_3(1-qu)}-\frac{up^4u^iq^{4i}}{q^3(qu;q)_4},\end{aligned}$$

as well as the next instance (a five fold sum)

$$u^i\longrightarrow\frac{p^5uq^{4i}}{q^4(q;q)_4(1-qu)}-\frac{up^5u^iq^{5i}}{q^4(qu;q)_5},$$

from which it is easy to guess the general substitution formula and to prove it by induction. This leads to the recursion

$$a_{n+1}(u)=\frac{p^ku}{q^{k-1}(q;q)_{k-1}(1-qu)}a_n(q^{k-1})-\frac{p^ku}{q^{k-1}(qu;q)_k}a_n(q^ku).$$

The rest of the proof is then as in the previous section.

4. Other patterns

The method we use is suitable to deal with other patterns as well. As an example, we consider words (and permutations in the limit) of length $\equiv 1 \bmod 4$ satisfying the pattern $(\le\le>>)^*$. A relatively straight forward computation leads to

$$a_1(u)=\frac{pu}{1-qu},$$

and for $n\ge1$

$$a_{n+1}(u)=\frac{p^3u}{q(q;q)_2(1-q^2u)}a_n(q^2)-\frac{p^4u}{q^3(q;q)_3(1-qu)}a_n(q^3)+\frac{p^4}{q^4(qu;q)_4}a_n(q^4u).$$

We translate that into a generating function,

$$f(u)=\sum_{n\ge1}z^{4n-3}a_n(u).$$

With this notation we get the functional equation

$$f(u)=\frac{zpu}{1-qu}+\frac{z^4p^3u}{q(q;q)_2(1-q^2u)}f(q^2)-\frac{z^4p^4u}{q^3(q;q)_3(1-qu)}f(q^3)+\frac{z^4p^4}{q^4(qu;q)_4}f(q^4u).\tag{2}$$

We must compute $f(q^2)$ and $f(q^3)$:

$$\begin{aligned}f(q^2)&=\frac{zpq^2}{1-q^3}+\frac{z^4p^3q^2}{q(q;q)_2(1-q^4)}f(q^2)-\frac{z^4p^4q^2}{q^3(q;q)_3(1-q^3)}f(q^3)\\&+\frac{z^4p^4}{q^4(q^3;q)_4}\bigg[\frac{zpq^6}{1-q^7}+\frac{z^4p^3q^6}{q(q;q)_2(1-q^8)}f(q^2)-\frac{z^4p^4q^6}{q^3(q;q)_3(1-q^7)}f(q^3)\\&+\frac{z^4p^4}{q^4(q^7;q)_4}\bigg[\frac{zpq^{10}}{1-q^{11}}+\frac{z^4p^3q^{10}}{q(q;q)_2(1-q^{12})}f(q^2)-\frac{z^4p^4q^{10}}{q^3(q;q)_3(1-q^{11})}f(q^3)\end{aligned}$$

$$+\frac{z^4p^4}{q^4(q^{11};q)_4}\Big[+\dots$$

So defining the auxiliary quantities

$$\alpha = q^2\sum_{n\ge 0}\frac{z^{4n+1}p^{4n+1}}{(q^3;q)_{4n+1}},$$
$$\beta = 1-\frac{q}{p}\sum_{n\ge 1}\frac{z^{4n}p^{4n}}{(q;q)_{4n-2}(1-q^{4n})},$$
$$\gamma = -\frac{1}{q(q;q)_3}\sum_{n\ge 1}\frac{z^{4n}p^{4n}}{(q^3;q)_{4n-3}},$$

we get

$$f(q^2)\beta = \alpha+\gamma f(q^3).$$

Similarly,

$$\begin{aligned} f(q^3) &= \frac{zpq^3}{1-q^4}+\frac{z^4p^3q^3}{q(q;q)_2(1-q^5)}f(q^2)-\frac{z^4p^4q^3}{q^3(q;q)_3(1-q^4)}f(q^3)\\ &+\frac{z^4p^4}{q^4(q^4;q)_4}\Big[\frac{zpq^7}{1-q^8}+\frac{z^4p^3q^7}{q(q;q)_2(1-q^9)}f(q^2)-\frac{z^4p^4q^7}{q^3(q;q)_3(1-q^8)}f(q^3)\\ &+\frac{z^4p^4}{q^4(q^8;q)_4}\Big[\frac{zpq^{11}}{1-q^{12}}+\frac{z^4p^3q^{11}}{q(q;q)_2(1-q^{13})}f(q^2)-\frac{z^4p^4q^{11}}{q^3(q;q)_3(1-q^{12})}f(q^3)\\ &+\frac{z^4p^4}{q^4(q^{12};q)_4}\Big[\dots, \end{aligned}$$

or, with

$$\delta = q^3\sum_{n\ge 0}\frac{z^{4n+1}p^{4n+1}}{(q^4;q)_{4n+1}},$$
$$\rho = \frac{q^2}{p(q;q)_2}\sum_{n\ge 1}\frac{z^{4n}p^{4n}}{(q^4;q)_{4n-4}(1-q^{4n+1})},$$
$$\sigma = \sum_{n\ge 0}\frac{z^{4n}p^{4n}}{(q;q)_{4n}},$$

we get

$$f(q^3)\sigma = \delta+\rho f(q^2).$$

Solving the system we find

$$f(q^2) = \frac{\alpha\sigma+\gamma\delta}{\beta\sigma-\gamma\rho},$$
$$f(q^3) = \frac{\alpha\rho+\beta\delta}{\beta\sigma-\gamma\rho}.$$

Now we can compute $f(1)$:

$$f(1) = \frac{zp}{1-q}+\frac{z^4p^3}{q(q;q)_2(1-q^2)}f(q^2)-\frac{z^4p^4}{q^3(q;q)_3(1-q)}f(q^3)$$

$$+\frac{z^4p^4}{q^4(q;q)_4}\bigg[\frac{zpq^4}{1-q^5}+\frac{z^4p^3q^4}{q(q;q)_2(1-q^6)}f(q^2)-\frac{z^4p^4q^4}{q^3(q;q)_3(1-q^5)}f(q^3)$$
$$+\frac{z^4p^4}{q^4(q^5;q)_4}\bigg[\frac{zpq^8}{1-q^9}+\frac{z^4p^3q^8}{q(q;q)_2(1-q^{10})}f(q^2)-\frac{z^4p^4q^8}{q^3(q;q)_3(1-q^9)}f(q^3)$$
$$+\frac{z^4p^4}{q^4(q^9;q)_4}\bigg[\dots,$$

or, with

$$\lambda=\sum_{n\ge0}\frac{z^{4n+1}p^{4n+1}}{(q;q)_{4n+1}},$$
$$\mu=\frac{1}{pq(q;q)_2}\sum_{n\ge0}\frac{z^{4n+4}p^{4n+4}}{(q;q)_{4n}(1-q^{4n+2})},$$
$$\nu=\frac{1}{q^3(q;q)_3}\sum_{n\ge1}\frac{z^{4n}p^{4n}}{(q;q)_{4n-3}},$$

$$f(1)=\lambda+\mu f(q^2)-\nu f(q^3).$$

We are not aiming to simplify this expression in the general case. However, we move straight to the limit $q\to1$, to study permutations. At this stage it seems necessary to make the variable z (and residue class $\pmod 4$) explicit: We write $H_1(z):=f(1)$. Then

$$H_1(z)=\lambda+\mu\frac{\alpha\sigma+\gamma\delta}{\beta\sigma-\gamma\rho}-\nu\frac{\alpha\rho+\beta\delta}{\beta\sigma-\gamma\rho},$$

with

$$\alpha=2\sum_{n\ge0}\frac{z^{4n+1}}{(4n+3)!},$$
$$\beta=1-\sum_{n\ge1}\frac{z^{4n}}{(4n-2)!(4n)},$$
$$\gamma=-\frac13\sum_{n\ge1}\frac{z^{4n}}{(4n-1)!},$$
$$\delta=6\sum_{n\ge0}\frac{z^{4n+1}}{(4n+4)!},$$
$$\rho=3\sum_{n\ge1}\frac{z^{4n}}{(4n-1)!(4n+1)},$$
$$\sigma=\sum_{n\ge0}\frac{z^{4n}}{(4n)!},$$
$$\lambda=\sum_{n\ge0}\frac{z^{4n+1}}{(4n+1)!},$$
$$\mu=\frac12\sum_{n\ge0}\frac{z^{4n+4}}{(4n)!(4n+2)},$$

$$\nu = \frac{1}{6}\sum_{n\ge1}\frac{z^{4n}}{(4n-3)!}.$$

All these series can be expressed by exponential functions as follows:

$$\begin{aligned}
\alpha &= \frac{1}{2z^2}\left[e^z + ie^{iz} - e^{-z} - ie^{-iz}\right],\\
\beta &= \frac{1}{4}\left[e^z + e^{iz} + e^{-z} + e^{-iz}\right] - \frac{z}{4}\left[e^z + ie^{iz} - e^{-z} - ie^{-iz}\right],\\
\gamma &= -\frac{z}{12}\left[e^z + ie^{iz} - e^{-z} - ie^{-iz}\right],\\
\delta &= -\frac{6}{z^3} + \frac{3}{2z^3}\left[e^z + e^{iz} + e^{-z} + e^{-iz}\right],\\
\rho &= \frac{3}{4}\left[e^z + e^{iz} + e^{-z} + e^{-iz}\right] - \frac{3}{4z}\left[e^z - ie^{iz} - e^{-z} + ie^{-iz}\right],\\
\sigma &= \frac{1}{4}\left[e^z + e^{iz} + e^{-z} + e^{-iz}\right],\\
\lambda &= \frac{1}{4}\left[e^z - ie^{iz} - e^{-z} + ie^{-iz}\right],\\
\mu &= \frac{z^3}{8}\left[e^z - ie^{iz} - e^{-z} + ie^{-iz}\right] - \frac{z^2}{8}\left[e^z - e^{iz} + e^{-z} - e^{-iz}\right],\\
\nu &= \frac{z^3}{24}\left[e^z - ie^{iz} - e^{-z} + ie^{-iz}\right].
\end{aligned}$$

Finally, this allows to express the generating function of interest as

$$H_1(z) = (1+i)\frac{-ie^{(1+i)z} - e^{(-1+i)z} + ie^{(-1-i)z} + e^{(1-i)z}}{e^{(1+i)z} + e^{(-1+i)z} + 4 + e^{(-1-i)z} + e^{(1-i)z}}.$$

Here is the list of the first coefficients (normalized by factorials, to make them the number of permutations satisfying the pattern $(\le\le>>)^n$, for $n = 0, 1, 2, \dots$):

$1, 6, 1456, 2020656, 9336345856, 108480272749056, 2664103110372192256,$
$122840808510269863827456, 9758611490955498257378246656,$
$1251231616578606273788469919481856, 245996119743058288132230759497577005056, \dots$.

This sequence is not in Sloane's encyclopedia [5, 6].

Now we sketch briefly what happens for the other residue classes $\pmod 4$. They can all be obtained from $f(u)$ by appropriate substitutions.

For an extra up step $\le$, we must compute $H_2(z) := \frac{z}{q}f(q)$. Starting from the functional equation (2) we obtain

$$\frac{1}{q}f(q) = \sum_{n\ge0}\frac{p^{4n+1}z^{4n+1}}{(q^2;q)_{4n+1}} + \frac{f(q^2)}{pq(q;q)_2}\sum_{n\ge1}\frac{p^{4n}z^{4n}}{(q^2;q)_{4n-4}(1-q^{4n-1})} - \frac{f(q^3)}{q^3(q;q)_3}\sum_{n\ge1}\frac{p^{4n}z^{4n}}{(q^2;q)_{4n-3}}.$$

If we let

$$\begin{aligned}
\lambda_1 &= \sum_{n\ge0}\frac{p^{4n+1}z^{4n+1}}{(q^2;q)_{4n+1}},\\
\varrho &= \frac{1}{pq(q^2;q)}\sum_{n\ge1}\frac{p^{4n}z^{4n}}{(q;q)_{4n-3}(1-q^{4n-1})},
\end{aligned}$$

$$\omega = \frac{1}{q^3(q;q)_3} \sum_{n\ge1} \frac{p^{4n}z^{4n}}{(q^2;q)_{4n-3}},$$

we obtain

$$\frac{1}{q}f(q) = \lambda_1 + \frac{\sigma\alpha+\gamma\delta}{\beta\alpha-\gamma\rho}\varrho - \frac{\rho\alpha+\beta\delta}{\beta\alpha-\gamma\rho}\omega.$$

Again, we concentrate on the limit $q \to 1$, to study permutations. So

$$H_2(z) = z\Big(\lambda_1 + \frac{\sigma\alpha+\gamma\delta}{\beta\alpha-\gamma\rho}\varrho - \frac{\rho\alpha+\beta\delta}{\beta\alpha-\gamma\rho}\omega\Big)$$

where

$$\lambda_1 = \sum_{n\ge0} \frac{z^{4n+1}}{(4n+1)!},$$

$$\varrho = \frac{1}{2}\sum_{n\ge1} \frac{z^{4n}}{(4n-3)!(4n-1)},$$

$$\omega = \frac{1}{6}\sum_{n\ge1} \frac{z^{4n}}{(4n-2)!}.$$

The three series ω, ϱ and λ_1 can be expressed by exponential functions as follows:

$$\lambda_1 = \frac{1}{4}[e^z - ie^{iz} - e^{-z} + ie^{-iz}],$$

$$\varrho = \frac{z^2}{8}[e^z - e^{iz} + e^{-z} - e^{-iz}) - \frac{z}{8}(e^z + ie^{iz} - e^{-z} - ie^{-iz}],$$

$$\omega = \frac{z^2}{24}[e^z - e^{iz} + e^{-z} - e^{-iz}].$$

This allows us to express the generating function of interest as follows:

$$H_2(z) = -i\frac{e^{(1+i)z} - e^{(1-i)z} - e^{(-1+i)z} + e^{(-1-i)z}}{e^{(1+i)z} + e^{(1-i)z} + 4 + e^{(-1+i)z} + e^{(-1-i)z}}.$$

The first few values are $1, 26, 10576, 20551376, 122087570176, 1733786041150976$, corresponding to $n = 2, 6, 10, 14, 18$.

For two extra up steps $\le\le$, we must compute $H_3(z) := \frac{z^2}{q(1+q)}f(q^2)$. In the limit $q \to 1$, this becomes

$$H_3(z) = 2\frac{e^z - e^{-z} + ie^{iz} - ie^{-iz}}{e^{(1+i)z} + e^{(1-i)z} + 4 + e^{(-1+i)z} + e^{(-1-i)z}}.$$

The first few values are $1, 71, 45541, 120686411, 908138776681$, corresponding to $n = 3, 7, 11, 15, 19$.

Finally, for 3 extra steps $\le\le>$ we compute

$$H_0(z) := 1 + \frac{z^3}{q^2(1+q)}f(q^2) - \frac{z^3}{q^3(1+q)(1+q+q^2)}f(q^3).$$

In the limit $q \to 1$, this becomes

$$H_0(z) = 2\frac{e^z + e^{-z} + e^{iz} + e^{-iz}}{e^{(1+i)z} + e^{(1-i)z} + 4 + e^{(-1+i)z} + e^{(-1-i)z}}.$$

The first few values are $1, 3, 413, 397023, 1402815833$, corresponding to $n = 0, 4, 8, 12, 16$.

Remark. Alternative expressions are as follows: If we define (following Carlitz)

$$\phi_i(z) = \sum_{n\ge0} \frac{z^{4n+i}}{(4n+i)!}, \qquad \text{for } i = 0, 1, 2, 3,$$

(these are the functions $\Phi_{4,i}(z)$ from the Introduction) and $N := \phi_0(z)^2 - \phi_1(z)\phi_3(z)$, then

$$\begin{aligned}
H_1(z) &= \frac{\phi_1(z)\phi_0(z) - \phi_2(z)\phi_3(z)}{N},\\
H_2(z) &= \frac{\phi_2(z)\phi_0(z) - \phi_3^2(z)}{N} = \frac{\phi_1^2(z) - \phi_3^2(z)}{2N},\\
H_3(z) &= \frac{\phi_3(z)}{N},\\
H_0(z) &= \frac{\phi_0(z)}{N}.
\end{aligned}$$

Carlitz and Scoville in [2, 3] have such results. However, in [2, p. 47] we see a result that is difficult to interpret; there is a drawing of an up-up-down-down permutation of a length $\equiv 1 \bmod 4$, but its length is denoted by $4n-1$; as a generating function the authors give

$$\frac{\phi_1(z)\phi_2(z) - \phi_0(z)\phi_3(z)}{N} = 2\frac{z^3}{3!} + 132\frac{z^7}{7!} + \dots,$$

but there is only one permutation of length 3 satisfying that pattern.

The equivalence of the two forms for $H_2(z)$ follow from the relation $\phi_1^2(z) + \phi_3^2(z) = 2\phi_0(z)\phi_2(z)$, given by [3, (3.12)].

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