SOME INFORMATION ABOUT THE BINOMIAL TRANSFORM

HELMUT PRODINGER

Department of Algebra and Discrete Mathematics
Technical University of Vienna, Austria

A few days ago I saw the paper [2]. I think that I can make some additional remarks that might not be totally useless for the *Fibonacci Community*!

Let (a_n) be a given sequence and $s_n = \sum_{k=0}^{n} \binom{n}{k} a_k$. Denoting the respective (ordinary) generating functions by $A(x)$ and $S(x)$, the paper in question mainly deals with the consequences of the formula

$$S(x) = \frac{1}{1-x} A\Big(\frac{x}{1-x}\Big). \tag{1}$$

Knuth [7] has introduced the *binomial transform* by

$$\hat{a}_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k,$$

and it is clear that this is the situation from above. But Philippe Flajolet and the present writer agreed about 10 years ago that there are just *exponential generating functions* hidden! They have a convolution formula

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k},$$

and upon choosing the b_k's to be all equal to 1, we have the old situation. So, denoting the exponential generating functions by $\bar{A}(x)$ and $\bar{S}(x)$, we have the even simpler formula $\bar{S}(x) = e^x \bar{A}(x)$. This can readily be inverted as $\bar{A}(x) = e^{-x} \bar{S}(x)$, whence

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} s_k.$$

These facts about exponential generating functions are of course folklore; one particular reference is [3].

Flajolet and Richmond [2], Schmid [8] and Kirschenhofer and the present writer [6] all made heavy use of (1). U.Schmid observed (among other writers) that an exponential generating function will be transformed into an ordinary generating function by the *Borel transform*.

Now the *generalization*

$$s_n = \sum_{k=0}^{n} \binom{n}{k} b^{n-k} c^k a_k$$

or

$$S(x) = \frac{1}{1-bx} A\Big(\frac{cx}{1-bx}\Big)$$

translates into

$$\bar{S}(x) = e^{bx} \bar{A}(cx).$$

Since

$$\bar{A}(x) = e^{-\frac{b}{c}x} \bar{S}(\frac{x}{c}),$$

we find the inversion formula

$$a_n = c^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b^{n-k} s_k \,.$$

The discussion in Theorem 2 becomes quite transparent, considering exponential generating functions. It is asked whenever we have

$$F_{pn+r} = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} s^k F_{qk+r},$$

where F_n denote Fibonacci numbers. The exponential generating function of the Fibonacci numbers F_n is

$$\frac{1}{\sqrt{5}} \left(e^{\alpha x} - e^{\beta x}\right),$$

with the usual $\alpha = (1+\sqrt{5})/2$ and $\beta = -1/\alpha = (1-\sqrt{5})/2$. More generally, the sequence F_{pn+r} leads to

$$\frac{1}{\sqrt{5}} \left(\alpha^r e^{\alpha^p x} - \beta^r e^{\beta^p x}\right).$$

The desired equality means in this language

$$\frac{1}{\sqrt{5}} \left(\alpha^r e^{\alpha^p x} - \beta^r e^{\beta^p x}\right) = e^{tx} \frac{1}{\sqrt{5}} \left(\alpha^r e^{\alpha^q s x} - \beta^r e^{\beta^q s x}\right),$$

from which we deduce the two equations

$$\alpha^p = t + \alpha^q s,$$
$$\beta^p = t + \beta^q s.$$

Subtracting them we see

$$s = \frac{\alpha^p - \beta^p}{\alpha^q - \beta^q} = \frac{F_p}{F_q}.$$

Further,

$$t = \alpha^p - \alpha^q \frac{\alpha^p - \beta^p}{\alpha^q - \beta^q} = (-1)^p \frac{\alpha^{q-p} - \beta^{q-p}}{\alpha^q - \beta^q} = (-1)^p \frac{F_{q-p}}{F_q}.$$

To justify this *equating of coefficients* we note that the functions $e^{\lambda x}$ are linearly independent; and the other possibility of grouping terms from the left and the right side would lead to the impossible equation $\alpha^r = -\beta^r$.

In [2] there is also the modification: What are the coefficients of

$$T(x) = A(\frac{cx}{1-bx})?$$

That means, what is the effect of deleting the first factor? We can answer this much more generally by considering (with an arbitrary complex parameter d)

$$T(x) = \frac{1}{(1-bx)^d} A(\frac{cx}{1-bx}).$$

In this derivation we will use the concept of *residues*, interesting *per se*.

We are using the substitution $w = \dfrac{cx}{1-bx}$ or $x = \dfrac{w}{c+bw}$. Therefore $1 - bx = \dfrac{c}{c+bw}$ and $dx = \dfrac{c}{(c+bw)^2} dw$ and thus

$$\begin{aligned}
t_n := [x^n]T(x) &= \frac{1}{2\pi i} \oint \frac{dx}{x^{n+1}} T(x) \\
&= \frac{1}{2\pi i} \oint \frac{dx}{x^{n+1}} \frac{(c+bw)^d}{c^d} A(w) \\
&= \frac{1}{2\pi i} \oint \frac{c\,dw}{(c+bw)^2} \frac{(c+bw)^{n+1}}{w^{n+1}} \frac{(c+bw)^d}{c^d} A(w) \\
&= c^{1-d} [w^n] (c+bw)^{n+d-1} A(w) \\
&= \sum_{k=0}^{n} \binom{n+d-1}{n-k} b^{n-k} c^k a_k .
\end{aligned}$$

Since

$$A(w) = \Big(\frac{c}{c+bw}\Big)^d T\Big(\frac{w}{c+bw}\Big),$$

we find in a similar way the inversion formula

$$a_n = c^{-n} \sum_{k=0}^{n} \binom{n+d-1}{n-k} (-1)^{n-k} b^{n-k} t_k .$$

The formula (1) is also useful to deal with Knuth's sum [5, eq. (7.6)]

$$u_n = \sum_{k=0}^{n} \binom{n}{k} \Big(-\frac{1}{2}\Big)^k \binom{2k}{k}.$$

Since

$$\begin{aligned} f(x) &:= \sum_{k\geq 0} \Big(-\frac{1}{2}\Big)^k \binom{2k}{k} x^k \\ &= \sum_{k\geq 0} \Big(-\frac{x}{2}\Big)^k \binom{2k}{k} \\ &= \frac{1}{\sqrt{1+2x}}, \end{aligned}$$

the generating function of the sequence u_n turns out to be

$$\frac{1}{1-x}\,\frac{1}{\sqrt{1+2\left(\dfrac{x}{1-x}\right)}} = \frac{1}{\sqrt{1-x^2}} = \sum_{n\geq 0} x^{2n}\binom{2n}{n}4^{-n}.$$

From this we see that $u_n = 2^{-n}\binom{n}{n/2}$ if n is even, and $u_n = 0$ otherwise.

I communicated this idea to Knuth, and he reported that Herbert Wilf came to this (or a similar) approach independently.

The formula (1) has also a *combinatorial interpretation*. If, for example, $A(x)$ enumerates certain words, so that a_n is the number of words of length n with a certain property, and we perform the operation "fill-in a new letter where and as often you want", then the new "language" has the generating function $S(x)$. For further details on such *combinatorial constructions* we refer to [1].

References

1. P.Flajolet, *Mathematical Methods in the Analysis of Algorithms and Data Structures*, in: Trends in Theoretical Computer Science, E.Börger, Ed., Computer Science Press, Rockville, Maryland (1988), 225–304.
2. P.Flajolet and L.B.Richmond, *Generalized Digital Trees and Their Difference-Differential Equations*, Random Structures and Algorithms **3** (1992), 305–320.
3. H.W.Gould, *Series Transformations for Finding Recurrences for Sequences*, Fibonacci Quarterly **28** (1990), 166–171.
4. P.Haukkanen, *Formal Power Series for Binomial Sums of Sequences of Numbers*, Fibonacci Quarterly **31** (1993), 28–31.
5. A.Jonassen and D.E.Knuth, *A Trivial Algorithm Whose Analysis isn't*, Journal of Computer and System Sciences **16** (1978), 301–322.
6. P.Kirschenhofer and H.Prodinger, *The Path Length of Random Skip Lists*, submitted (1993).
7. D.E.Knuth, *The Art of Computer Programming Vol.3*, Addison-Wesley, Reading, MA, 1973.
8. U.Schmid, *The Average CRI-Length of a Tree Collision Resolution Algorithm in Presence of Multiplicity-Dependent Capture Effects*, Lecture Notes in Computer Science **623** (1992), 223–234.

AMS classification: 05A10

TU Vienna
Wiedner Hauptstrasse 8–10
A-1040 Vienna
Austria
E-mail address: proding@email.tuwien.ac.at