# ON THE MOMENTS OF A DISTRIBUTION DEFINED BY THE GAUSSIAN POLYNOMIALS

HELMUT PRODINGER

Di Bucchianico [1] has considered the probability distribution defined by the (probability) generating function

$$F_{m,n}(q) = \frac{\begin{bmatrix} m+n \\ m \end{bmatrix}_q}{\binom{m+n}{m}},$$

with the Gaussian polynomials

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \frac{(1-q)(1-q^2)\dots(1-q^{m+n})}{(1-q)(1-q^2)\dots(1-q^m)\cdot(1-q)(1-q^2)\dots(1-q^n)}.$$

He discussed several methods to compute the moments of this distribution.

In this short note, I want to draw the attention of the reader to another (potentially superior) method that is due to Panny [2]. This method first computes the *cumulants* and translates them into the *moments* by a standard formula. This yields *explicit* formulæ for all the moments.

Note that

$$F_{m,n}(q) = \frac{g_1(q)\dots g_{m+n}(q)}{g_1(q)\dots g_m(q)\cdot g_1(q)\dots g_n(q)}$$

with

$$g_k(q) = \frac{1-q^k}{k(1-q)},$$

where $g_k(q)$ is a probability generating function of a random variable $X_k$. The function $\log g_k(e^{it})$ is the generating function of the cumulants[1]. Panny has computed that as

$$\log g_k(e^{it}) = \frac{(k-1)it}{2} + \sum_{j\ge1} \frac{B_{2j}}{2j}(k^{2j}-1)\frac{(it)^{2j}}{(2j)!}, \qquad |t| < \frac{2\pi}{k},$$

where $B_i$ denotes the $i$th Bernoulli number.

Reading off coefficients we find the cumulants of $X_k$ as

$$\kappa_1 = \frac{k-1}{2}, \qquad \kappa_{2r} = \frac{(k^{2r}-1)B_{2r}}{2r}, \qquad \kappa_{2r+1} = 0, \qquad r = 1,2,3,\dots.$$

Since

$$\log F_{m,n}(e^{it}) = \sum_{k=1}^{m+n} \log g_k(e^{it}) - \sum_{k=1}^{m} \log g_k(e^{it}) - \sum_{k=1}^{n} \log g_k(e^{it}),$$

the cumulants $\kappa_r^{m,n}$ are obtained by summing the $\kappa_r$'s.

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[1]We write $\log x$ for the (natural) logarithm with base $e$.

So we get

$$\begin{aligned}
\kappa_1^{m,n} &= \frac{\binom{m+n}{2}}{2} - \frac{\binom{m}{2}}{2} - \frac{\binom{n}{2}}{2} = \frac{mn}{2},\\
\kappa_{2r}^{m,n} &= \frac{B_{2r}}{2r(2r+1)}\big(B_{2r+1}(m+n+1) - B_{2r+1}(m+1) - B_{2r+1}(n+1)\big),\\
\kappa_{2r+1}^{m,n} &= 0.
\end{aligned}$$

Here is a little list:

$$\begin{aligned}
\kappa_1^{m,n} &= \frac{mn}{2},\\
\kappa_2^{m,n} &= \frac{mn(m+n+1)}{12},\\
\kappa_4^{m,n} &= -\frac{mn(m+n+1)\big(m(m+1)+mn+n(n+1)\big)}{120}.
\end{aligned}$$

The formula to compute the moments $\mu_r$ from this is

$$\mu_r = \sum_{\pi_1+2\pi_2+\dots+r\pi_r=r} \Big(\frac{\kappa_1}{1!}\Big)^{\pi_1}\Big(\frac{\kappa_2}{2!}\Big)^{\pi_2}\dots\Big(\frac{\kappa_r}{r!}\Big)^{\pi_r}\frac{r!}{\pi_1!\pi_2!\dots\pi_r!},$$

which is a finite sum since $\pi_k \in \{0,1,\dots,r\}$. Hence

$$\begin{aligned}
\mu_1 &= \frac{mn}{2},\\
\mu_2 &= \kappa_1^2 + \kappa_2 = \frac{mn(m+n+3mn+1)}{12},\\
\mu_3 &= \kappa_1^3 + 3\kappa_1\kappa_2 + \kappa_3 = \frac{m^2(m+1)n^2(n+1)}{8}, \qquad \&c.
\end{aligned}$$

We finish by mentioning that, as in [2], one could also get asymptotic formulæ.

## References

[1] A. Di Bucchianico. Combinatorics, computer algebra, and the Wilcoxon–Mann–Whitney test. *Journal of Statistical Planning and Inference*, 79:349–364, 1999.

[2] W. Panny. A note on the higher moments of the expected behavior of straight insertion sort. *Information Processing Letters*, 22:175–177, 1986.

Helmut Prodinger, The John Knopfmacher Centre for Applicable Analysis and Number Theory, Department of Mathematics, University of the Witwatersrand, P. O. Wits, 2050 Johannesburg, South Africa, email: helmut@gauss.cam.wits.ac.za,
homepage: http://www.wits.ac.za/helmut/index.htm