THE EIGENVECTORS OF THE RIGHT–JUSTIFIED PASCAL TRIANGLE

DAVID CALLAN AND HELMUT PRODINGER

Let $R = \left(\binom{i-1}{n-j}\right)_{1\le i,j\le n}$ denote the $n \times n$ matrix formed by right justifying the first n rows of Pascal's triangle and let a denote the golden ratio $(1+\sqrt{5}\,)/2$. We will show that the eigenvalues of R are $\lambda_j = (-1)^{n-j}a^{2j-n-1}$, $1 \le j \le n$, (as conjectured in [1]), with corresponding eigenvectors $\mathbf{u}_j = (u_{ij})_{1\le i\le n}$ where

$$u_{ij} = (-a)^{n-j}\sum_{k=1}^{j}(-1)^{i-k}\binom{i-1}{k-1}\binom{n-i}{j-k}a^{2k-i-1}.$$

Let $U = (u_{ij})_{1\le i,j\le n}$ denote the matrix whose columns are the eigenvectors $(\mathbf{u}_j)_{j=1}^{n}$. For example, when $n=5$,

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 2 & 1\\ 0 & 1 & 3 & 3 & 1\\ 1 & 4 & 6 & 4 & 1 \end{pmatrix} \quad\text{and}\quad U = \begin{pmatrix} a^4 & -4\,a^3 & 6\,a^2 & -4\,a & 1\\ -a^3 & 3\,a^2-a^4 & -3\,a+3\,a^3 & 1-3\,a^2 & a\\ a^2 & -2\,a+2\,a^3 & 1-4\,a^2+a^4 & 2\,a-2\,a^3 & a^2\\ -a & 1-3\,a^2 & 3\,a-3\,a^3 & 3\,a^2-a^4 & a^3\\ 1 & 4\,a & 6\,a^2 & 4\,a^3 & a^4 \end{pmatrix}.$$

Remarkably, the left eigenvectors of R (the eigenvectors of R^t) are the *rows* of U. This follows from the fortuitous fact that U^2 is a scalar matrix, specifically $U^2 = (1+a^2)^{n-1}I_n$. To explain, let $V = (1+a^2)^{-(n-1)/2}U$ so that V is also a matrix of eigenvectors for R and $V^2 = I_n$. Taking transposes in $V^{-1}\,R\,V = \operatorname{diag}(\lambda_j)_{j=1}^{n}$ yields $W^{-1}\,R^t\,W = \operatorname{diag}(\lambda_j)_{j=1}^{n}$ with $W = \left(V^{-1}\right)^t$. In words, if a matrix R has a full set of eigenvectors forming the columns of a matrix V, then the columns of $\left(V^{-1}\right)^t$ are the corresponding eigenvectors for R^t. But here $\left(V^{-1}\right)^t$ is simply V^t. So the rows of V, as well as the rows of U (a scalar multiple of V), are eigenvectors for R^t.

Our proofs that $R\mathbf{u}_j = \lambda_j\mathbf{u}_j$, $1 \le j \le n$, and $U^2 = (1+a^2)^{n-1}I_n$ illustrate the power of generating functions [2]. Consider the generating function $U_i(z) = z(z-a)^{n-i}(az+1)^{i-1}$.

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It is immediate that

$$U_i(z) = \sum_{j=1}^{n} u_{ij} z^j.$$

We must prove that $(R\mathbf{u}_j)_i = (\lambda_j \mathbf{u}_j)_i$ for all i with $1 \le i \le n$. Now

$$\begin{aligned}
(R\mathbf{u}_j)_i &= [z^j] \sum_{k=1}^{n} R_{ik} U_k(z) \\
&= [z^j] \sum_{k=1}^{n} \binom{i-1}{n-k} z(z-a)^{n-k}(az+1)^{k-1} \\
&= [z^{j-1}](az+1)^{n-i} \sum_{k=1}^{n} \binom{i-1}{n-k} (z-a)^{n-k}(az+1)^{i-1-(n-k)} \\
&= [z^{j-1}](az+1)^{n-i}\big((z-a)+(az+1)\big)^{i-1} \\
&= [z^{j-1}](az+1)^{n-i}\big((1+a)z+(1-a)\big)^{i-1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\lambda_j \mathbf{u}_j)_i &= [z^j] \sum_{j=1}^{n} (-1)^{n+j} a^{2j-n-1} u_{ij} z^j \\
&= [z^j](-1)^n a^{-n-1} \sum_{j=1}^{n} u_{ij}(-za^2)^j \\
&= [z^j](-1)^n a^{-n-1} U_i(-a^2 z) \\
&= [z^j](-1)^n a^{-n-1}(-a^2 z)(-a^2 z - a)^{n-i}(-a^3 z + 1)^{i-1} \\
&= [z^{j-1}](az+1)^{n-i}\left(a^2 z - \frac{1}{a}\right)^{i-1},
\end{aligned}$$

and the claim follows since $a^2 = 1 + a$, and $-\frac{1}{a} = 1 - a$.

Next, we examine the (i,k) entry of U^2.

$$\begin{aligned}
(U^2)_{ik} &= \sum_{j=1}^{n} [x^{k-1}](x-a)^{n-j}(ax+1)^{j-1} \cdot [z^{j-1}](z-a)^{n-i}(az+1)^{i-1} \\
&= [x^{k-1}] \sum_{j=1}^{n} [z^{j-1}](x-a)^{n-j}(ax+1)^{j-1}(z-a)^{n-i}(az+1)^{i-1} \\
&= [x^{k-1}](x-a)^{n-1} \sum_{j=1}^{n} [z^{j-1}] \left(\frac{ax+1}{x-a}\right)^{j-1} (z-a)^{n-i}(az+1)^{i-1}
\end{aligned}$$

$$
\begin{aligned}
&= [x^{k-1}](x-a)^{n-1}\sum_{j=1}^{n}[z^{j-1}]\Big(z\frac{ax+1}{x-a}-a\Big)^{n-i}\Big(az\frac{ax+1}{x-a}+1\Big)^{i-1}\\
&= [x^{k-1}](x-a)^{n-1}\Big(\frac{ax+1}{x-a}-a\Big)^{n-i}\Big(a\frac{ax+1}{x-a}+1\Big)^{i-1}\\
&= [x^{k-1}](ax+1-ax+a^2)^{n-i}(a^2x+a+x-a)^{i-1}\\
&= [x^{k-1}](1+a^2)^{n-i}x^{i-1}(1+a^2)^{i-1}\\
&= [x^{k-i}](1+a^2)^{n-1}\\
&= (1+a^2)^{n-1}\delta_{ki},
\end{aligned}
$$

as desired.

Finally, a mild generalization. Let x be an indeterminate and let $R(x)$ be the $n\times n$ matrix with (i,j) entry $\binom{i-1}{n-j}x^{i+j-n-1}$. Then the eigenvalues and eigenvectors of $R(x)$ are precisely as above but with a a root of a^2-ax-1 rather than of a^2-a-1.

References

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David Callan, Department of Statistics, University of Wisconsin-Madison, 1210 W. Dayton Street, Madison, WI 53706-1693, U.S.A., email: callan@stat.wisc.edu

Helmut Prodinger, The John Knopfmacher Centre for Applicable Analysis and Number Theory, Department of Mathematics, University of the Witwatersrand, P. O. Wits, 2050 Johannesburg, South Africa, email: helmut@gauss.cam.wits.ac.za