ON CANTOR'S SINGULAR MOMENTS

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Abstract. We evaluate a constant explicitly, thereby answering a question raised in [1].

In problem **10621** of the American Mathematical Monthly, Cantor's singular moments J_n were to be computed. In the published answers [1] they come out as

$$J_n = \frac{2}{3(n+1)} \sum_{j=0}^{n} \binom{n+1}{j} \frac{B_j}{3\cdot 2^{j-1}-1} \quad \text{for} \quad n \ge 1$$

and $J_0 = 1$, with Bernoulli numbers B_n.

The editor asked, whether it is possible to compute

$$J_{-1} = \sum_{n\ge 0} J_n$$

exactly.

The purpose of this note is to do that. In [5] we considered a similar problem, and the gentle reader is invited to consult this paper for more background about the technique, as well as [6, 4] for more information about the Cantor distribution.

Following the method described in [3], we can write J_n as a contour integral viz.

$$\begin{aligned} J_n &= \frac{2}{3(n+1)} \cdot \frac{1}{2\pi i} \int_{-\frac12 - i\infty}^{-\frac12 + i\infty} \frac{\Gamma(n+2)\Gamma(1-s)}{\Gamma(n+2-s)} \frac{\zeta(1-s)}{3\cdot 2^{s-1}-1}\, ds \\ &= \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac12 - i\infty}^{-\frac12 + i\infty} \frac{\Gamma(n+1)\Gamma(1-s)}{\Gamma(n+2-s)} \frac{\zeta(1-s)}{3\cdot 2^{s-1}-1}\, ds. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n=0}^{N} J_n = J_0 &+ \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac12 - i\infty}^{-\frac12 + i\infty} \frac{\Gamma(N+2)\Gamma(1-s)}{\Gamma(N+2-s)s} \frac{\zeta(1-s)}{3\cdot 2^{s-1}-1}\, ds \\ &- \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac12 - i\infty}^{-\frac12 + i\infty} \frac{\Gamma(2)\Gamma(1-s)}{\Gamma(2-s)s} \frac{\zeta(1-s)}{3\cdot 2^{s-1}-1}\, ds. \end{aligned}$$

From this form, one could even compute the asympotics as $N \to \infty$. However, here, we only have to note that the first integral is of order $N^{1-\log_2 3}$, which means that it

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goes to zero. Consequently

$$\begin{aligned}
\sum_{n\ge0} J_n &= 1-\frac23\cdot\frac{1}{2\pi i}\int_{-\frac12-i\infty}^{-\frac12+i\infty}\frac{1}{(1-s)s}\frac{\zeta(1-s)}{3\cdot2^{s-1}-1}\,ds\\
&=\frac43+\frac23\cdot\frac{1}{2\pi i}\int_{\frac32-i\infty}^{\frac32+i\infty}\frac{1}{s(s-1)}\frac{\zeta(s)}{3\cdot2^{-s}-1}\,ds\\
&=\frac43+\frac23\sum_{k,m\ge1}3^{-k}\cdot\frac{1}{2\pi i}\int_{\frac32-i\infty}^{\frac32+i\infty}\frac{1}{s(s-1)}\Big(\frac{2^k}{m}\Big)^s\,ds.
\end{aligned}$$

The last step was by using the Dirichlet series for $\zeta(s)$ and the geometric series, both valid for $\Re s=\frac32$. A simple application of residue calculus, as it is often used in the context of the Mellin–Perron summation formula (see [7, 2]) evaluates the integrals inside the summation:

$$\frac{1}{2\pi i}\int_{\frac32-i\infty}^{\frac32+i\infty}\frac{1}{s(s-1)}t^s\,ds=\begin{cases}t-1 & \text{for } t\ge1\\ 0 & \text{for } t<1\end{cases}.$$

Therefore

$$\begin{aligned}
\sum_{n\ge0} J_n &= 1+\frac23\sum_{k\ge1}\sum_{1\le m\le2^k}3^{-k}\Big(\frac{2^k}{m}-1\Big)\\
&=1+\frac23\sum_{k\ge1}\Big(\frac23\Big)^kH_{2^k}-\frac23\sum_{k\ge1}\Big(\frac23\Big)^k\\
&=-\frac13+\frac23\sum_{k\ge1}\Big(\frac23\Big)^kH_{2^k}=3.36465\ 07281\ 00925\ 16083\ 89349\ 6289\ldots,
\end{aligned}$$

with harmonic numbers $H_n=\sum_{1\le k\le n}\frac1k$.

References

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