ON CARLITZ COMPOSITIONS

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Abstract. This paper deals with Carlitz compositions of natural numbers (adjacent parts have to be different). The following parameters are analyzed: Number of parts, number of equal adjacent parts in ordinary compositions, largest part, Carlitz compositions with zeros allowed (correcting an erroneous formula from Carlitz). It is also briefly demonstrated that so–called 1–compositions of a natural number can be treated in a similar style.

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1. Introduction

A restricted composition of a natural number n in the sense of Carlitz [4], which we shall call a *Carlitz composition,* is defined to be a composition

$$n = a_1 + a_2 + \cdots + a_k \quad \text{such that} \quad a_i \neq a_{i+1} \text{ for } i = 1, \ldots, k-1.$$

We refer to n as the *size* and to k as the *number of parts* of the composition.

Observe that there are 2^{n-1} unrestricted compositions of the integer n with generating function $1/(1 - z/(1-z))$.

Let $c(n)$ denote the number of Carlitz compositions of n. In [4], Carlitz found the generating function

$$C(z) := \sum_{n \geq 0} c(n) z^n.$$

Since we are going to compute several related parameters we find it useful to rederive his result in a streamlined way, using a method that has appeared for example in [6] under the nickname "adding a new slice." We proceed from a Carlitz composition with k parts to one with $k+1$ parts by allowing a_{k+1} to be any number and then subtracting the forbidden case $a_{k+1} = a_k$. In terms of generating functions this reads as follows. Let $f_k(z, u)$ be the generating function of those Carlitz compositions with k parts where the coefficient of $z^n u^j$ refers to size n and last part $a_k = j$. Then

$$f_{k+1}(z, u) = f_k(z, 1) \frac{zu}{1 - zu} - f_k(z, zu) + \delta_{k,0} \quad \text{for } k \geq 0, \quad f_0(z, u) = 1. \qquad (*)$$

The first term means that we forget the labelling of the last part ($u := 1$) and add any term, together with a labelling by u, and the second one means that we subtract the forbidden term, which is a repetition of the previous last part. Introducing $F(z,u) := \sum_{k\ge1} f_k(z,u)$ and summing on $k \ge 0$, we get

$$F(z,u) = F(z,1)\frac{zu}{1-zu} + \frac{zu}{1-zu} - F(z,zu).$$

This functional equation can now be *iterated* and gives

$$F(z,1) = \sigma(z) + F(z,1)\,\sigma(z),$$

with

$$\sigma(z) = \sum_{j\ge1} \frac{z^j(-1)^{j-1}}{1-z^j}.$$

Since $C(z) = 1 + F(z,1)$, we find the formula of Carlitz,

$$C(z) = \frac{1}{1-\sigma(z)}.$$

Here are the first few values:

$$1 + z + z^2 + 3z^3 + 4z^4 + 7z^5 + 14z^6 + 23z^7 + 39z^8 + 71z^9 + 124z^{10} + \dots;$$

it is sequence $A003242$ in [14].

In his paper, Carlitz noted only that the radius of convergence of $C(z)$ is at least $\frac{1}{2}$. Now we can go further than that and notice that there is a dominant pole ρ, which is the unique real solution in the interval $[0,1]$ of the equation $\sigma(z) = 1$. Numerically we find $\rho = 0.571349\dots$ The other poles are further away, which can proved by Rouché's theorem very much as in [6,9,11,12,13].

Consequently, in a neighbourhood of $z = \rho$,

$$C(z) \sim \frac{A}{1-z/\rho}, \quad \text{with} \quad A = \frac{1}{\rho\sigma'(\rho)} = 0.456387\dots$$

Therefore,

$$c(n) \sim A\rho^{-n} = 0.456387 \cdot (1.750243)^n.$$

2. The number of parts

Now we are interested to know how many parts a (random) Carlitz composition of size n has (on the average).

We will use another variable, w, to label the number of parts. The functional recursion $(*)$ is of course our starting point. Introducing $G(z,u,w) := \sum_{k\ge1} w^k f_k(z,u)$, we find by multiplying $(*)$ by w^{k+1} and summing over $k \ge 0$,

$$G(z,u,w) = wG(z,1,w)\frac{zu}{1-zu} + w\frac{zu}{1-zu} - wG(z,zu,w).$$

Iterating this as before we find

$$G(z,1,w)=\frac{\tau(z,w)}{1-\tau(z,w)},$$

with

$$\tau(z,w)=\sum_{j\geq 1}\frac{z^j w^j(-1)^{j-1}}{1-z^j}.$$

$n\backslash k$	1	2	3	4	5	6	7	8
1	1							
2	1							
3	1	2						
4	1	2	1					
5	1	4	2					
6	1	4	7	2				
7	1	6	9	6	1			
8	1	6	15	14	3			
9	1	8	21	24	15	2		
10	1	8	28	46	30	10	1	
11	1	10	35	66	68	30	4	
12	1	10	46	100	119	76	24	2

TABLE 1. Carlitz compositions by size and number of parts

To compute the average value, we need the function

$$\overline{G}(z):=\frac{\partial}{\partial w}G(z,1,w)\Big|_{w=1},$$

for which we easily find

$$\overline{G}(z)=\frac{\mu(z)}{(1-\sigma(z))^2}\quad\text{with}\quad \mu(z)=\sum_{j\geq 1}\frac{jz^j(-1)^{j-1}}{1-z^j}.$$

Consequently,

$$\overline{G}(z)\sim\frac{B}{(1-z\rho)^2},\quad\text{with}\quad B=\mu(\rho)A^2=0.159996.$$

Therefore

$$[z^n]\overline{G}(z)\sim B\,n\,\rho^{-n},$$

and thus the average number of parts in a Carlitz composition of size n is asymptotic to

$$\frac{B\,n\,\rho^{-n}}{A\,\rho^{-n}}=0.350571\cdot n.$$

By contrast, an unrestricted composition of n has $\frac{n+1}{2}$ parts on average.

3. Counting adjacent equal parts

In this section we consider all compositions of n and count the number of adjacent equal parts. The original Carlitz case is then equivalent to compositions where this count gives zero.

Again our starting point will be recursion ($*$); we will use a variable v to count the number of adjacent equal parts. ($v = 0$ means the Carlitz case.)

We have by the same reasoning as before

$$f_{k+1}(z,u,v) = f_k(z,1,v)\frac{zu}{1-zu} - (1-v)f_k(z,zu,v) + (1-v)\delta_{k,0} \quad \text{for } k \geq 0,$$

and $f_0(z,u,v) = 1$. Defining $F(z,u,v) := \sum_{k\geq 1} f_k(z,u,v)$ and summing we get

$$F(z,u,v) = F(z,1,v)\frac{zu}{1-zu} + \frac{zu}{1-zu} - (1-v)F(z,zu,v).$$

Iterating as before we find for the generating function of interest $1 + F(z,1,v)$ that

$$1 + F(z,1,v) = \frac{1}{1-\sigma(z,v)} \quad \text{with} \quad \sigma(z,v) = \sum_{j\geq 1} \frac{z^j(v-1)^{j-1}}{1-z^j}.$$

Observe that $v := 0$ gives the generating function of the Carlitz compositions and $v := 1$ gives the generating function $\frac{1-z}{1-2z}$ of all compositions, as it should.

If we are thus interested in the generating function of compositions with exactly m adjacent equal parts, then we must extract the coefficient of v^m, which we might do by Taylor's formula as

$$[v^m]\frac{1}{1-\sigma(z,v)} = \frac{1}{m!}\frac{\partial^m}{\partial v^m}\frac{1}{1-\sigma(z,v)}\bigg|_{v=0}.$$

For instance, we get for $m = 1$

$$\frac{\sigma'(z,0)}{(1-\sigma(z))^2},$$

the derivative being w.r.t. v. For general m we get something of the form

$$\frac{p_m(z)}{(1-\sigma(z))^{m+1}},$$

with a function $p_m(z)$ that is built up from derivatives of the function $\sigma(z,v)$. Consequently, for fixed m and $n \to \infty$ the number of compositions with exactly m equal adjacent parts has the following asymptotic behaviour

$$C_m\, n^m\, \rho^{-n}$$

with explicitly computable constants C_m. The first few are

$$C_0 = 0.4563,$$
$$C_1 = 0.0482,$$
$$C_2 = 0.0025.$$

Also, we can determine the average number of equal adjacent parts by differentiation w.r.t. v, followed by setting $v=1$, which yields

$$\frac{(1-z)^2}{(1-2z)^2}\frac{z^2}{1-z^2}=\frac{z^2(1-z)}{(1+z)(1-2z)^2}.$$

For $n\ge 1$ the coefficient of z^n therein is

$$\frac{1}{12}(n+1)2^n-\frac{1}{18}2^n+\frac{2}{9}(-1)^n,$$

which gives (upon division by 2^{n-1}) our average as

$$\frac{3n+1}{18}+O(2^{-n}).$$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
1	1											
2	1	1										
3	3	0	1									
4	4	3	0	1								
5	7	6	2	0	1							
6	14	7	8	2	0	1						
7	23	20	10	8	2	0	1					
8	39	42	22	13	9	2	0	1				
9	71	72	58	28	14	10	2	0	1			
10	124	141	112	72	33	16	11	2	0	1		
11	214	280	219	150	92	36	18	12	2	0	1	
12	378	516	466	311	189	112	40	20	13	2	0	1

TABLE 2. Compositions $C(n,k)$ by size and number of adjacent equal parts

REMARK. We notice formulæ such as

$$\begin{aligned}
&C(n,n-1)=1, \quad n\ge 1,\\
&C(n,n-2)=0, \quad n\ge 3,\\
&C(n,n-3)=2, \quad n\ge 5,\\
&C(n,n-4)=n+1, \quad n\ge 7,\\
&C(n,n-5)=2n-4, \quad n\ge 9,\\
&C(n,n-6)=4n-8, \quad n\ge 11,\\
&C(n,n-7)=\frac{n^2+15n-102}{2}, \quad n\ge 13,\dots
\end{aligned}$$

They are all easy to prove by considering

$$\frac{1}{1-\sigma(zv,v^{-1})}$$

and looking for the coefficient of a fixed power of v.

4. The largest part in Carlitz compositions

For ordinary partitions, the statistic "largest part of a composition" has obtained a lot of attention, [5,10]. Now we want to sketch the analogous analysis for the case of Carlitz compositions.

Let us first consider the generating function(s) where all parts are less than or equal to h. Then essentially the same idea as in $(*)$ works, except that we only use a factor

$$(zu) + \cdots + (zu)^h = \frac{zu(1-(zu)^h)}{1-zu}$$

instead of the full geometric series. This results finally in

$$1 + F_h(z,1) = \frac{1}{1-\sigma_h(z)},$$

with

$$\sigma_h(z) = \sum_{j\ge 1} \frac{\big(z^j - z^{j(h+1)}\big)(-1)^{j-1}}{1-z^j}.$$

The dominant pole ρ_h is now the (positive real) solution of the equation $\sigma_h(z) = 1$. It is clear that ρ_h tends to ρ, but we have to determine how fast. We will use the "bootstrapping method" from [11].

Now, around $z = \rho$ we have the approximate equation

$$1 \approx \sigma(z) - \rho^{h+1}.$$

Using Taylor's theorem and setting $\rho_h = \rho(1+\varepsilon_h)$, we arrive at

$$0 \approx \rho\varepsilon_h\sigma'(\rho) - \rho^{h+1},$$

or

$$\varepsilon_h \approx \frac{1}{\sigma'(\rho)}\rho^h.$$

Therefore the number of Carlitz compositions with largest part $\le h$ is approximated by

$$\frac{1}{\rho_h\sigma'(\rho_h)}\rho_h^{-n} \approx \frac{1}{\rho\sigma'(\rho)}\rho^{-n}(1+\varepsilon_h)^{-n}.$$

Consequently the probability that a Carlitz composition has largest part $\le h$ is approximated by

$$(1+\varepsilon_h)^{-n} \approx \Big(1 - \frac{1}{\sigma'(\rho)}\rho^h\Big)^n.$$

For the probability that the largest part is $> h$ we have then approximately

$$1 - \Big(1 - \frac{1}{\sigma'(\rho)}\rho^h\Big)^n,$$

and to get the desired average value E_n we must sum this up over $h \ge 0$. The next step is to use the exponential approximation $(1-a)^n \approx e^{-an}$;

$$E_n \approx \sum_{h\ge 0} \Big(1 - e^{-n\rho^h/\sigma'(\rho)}\Big).$$

But this quantity is quite well studied [5] (we might even set $N := n/\sigma'(\rho)$ for the moment to make it look closer to already existing formulæ).

The answer is

$$E_n \sim \log_{1/\rho} N - \frac{\gamma}{\log \rho} + \frac{1}{2} + \delta(\log_{1/\rho} N),$$

with a certain periodic function $\delta(x)$ that has period 1, mean 0, and small amplitude.

Rewriting this we find

$$E_n \sim \log_{1/\rho} n - \log_{1/\rho} \sigma'(\rho) - \frac{\gamma}{\log \rho} + \frac{1}{2} + \overline{\delta}(\log_{1/\rho} n),$$

where $\overline{\delta}(x) = \delta(x - \log_{1/\rho} \sigma'(\rho))$, which has the same properties as $\delta(x)$. The numerical constant is $-\log_{1/\rho} \sigma'(\rho) - \frac{\gamma}{\log \rho} + \frac{1}{2} = -0.870252$.

It might be of interest for the reader to learn that Xavier Gourdon studied *largest components in combinatorial structures* in great generality [8]. From this treatment it seems that almost the distributions in our paper are *asymptotically Gaussian*, although we have not performed a rigorous analysis. The distribution of the largest part (section 4) follows a *double exponential law.* This holds under very general conditions, e. g. for *unrestricted compositions*, compare [8; p. 190ff]. The paper [3] should also be mentioned in this context.

5. Carlitz Compositions with zeros

For ordinary compositions it is meaningless to allow the a_i's to be zero, since then there would be infinitely many compositions for each n. However, in the context of Carlitz compositions, it *makes* sense, since one can have at most $n+1$ zeros, so the number $\overline{c}(n)$ of Carlitz compositions with zeros allowed is meaningful.

In [4] Carlitz gave an erroneous formula for the generating function

$$\overline{C}(z) = \sum_{n\ge 0} \overline{c}(n) z^n$$

viz.

$$\overline{C}(z) = \frac{1}{1-\overline{\sigma}(z)} \quad \text{with} \quad \overline{\sigma}(z) = (1-z)\sum_{j\ge 1} \frac{z^{2j-1}}{(1-z^{2j-1})(1-z^{2j})}.$$

A simple rearrangement shows that $C(z) = \overline{C}(z)$, so this cannot be correct. Here is a brief explanation of the flaw in the derivation.

Carlitz gives *mutatis mutandis*

$$f_{k+1}(z,u)=f_k(z,1)\frac{1}{1-zu}-f_k(z,zu)+\delta_{k,0}\quad \text{for } k\ge 0,\quad f_0(z,u)=1$$

and

$$F(z,u)=F(z,1)\frac{1}{1-zu}+\frac{1}{1-zu}-F(z,zu),$$

as in section 1, which is still correct, but then he iterates that, which is prohibited, because of a problem with the *constant term.*

Here is a corrected version: We dissect the set of compositions into those with last part ≥ 1 (counted by $g_k(z,u)$) and those with last part $=0$ (counted by $h_k(z)$). Clearly, $h_k(z)=g_{k-1}(z,1)$.

Then the usable recursion is (for $k\ge 1$)

$$g_{k+1}(z,u)=g_k(z,1)\frac{zu}{1-zu}+h_k(z)\frac{zu}{1-zu}-g_k(z,zu).$$

Denoting $G(z,u)=\sum_{k\ge1}g_k(z,u)$ and summing up we get

$$G(z,u)=2\frac{zu}{1-zu}+2G(z,1)\frac{zu}{1-zu}-G(z,zu).$$

This version *is* now amenable to iteration, and consequently we get (with the function $\sigma(z)$ from the introduction)

$$G(z,1)=\frac{2\sigma(z)}{1-2\sigma(z)}.$$

Therefore

$$\overline{C}(z)=1+2G(z,1)=\frac{1+2\sigma(z)}{1-2\sigma(z)}.$$

Again, there is a dominant singularity $\overline{\rho}$, which is the solution in the interval $[0,1]$ of the equation $\sigma(z)=\frac{1}{2}$.

Numerically we find $\overline{\rho}=0.386960$. This is also in contrast to Carlitz's comment that is should be close to $\frac{1}{2}$.

Thus

$$\overline{c}(n)\sim\frac{1+2\sigma(\overline{\rho})}{2\overline{\rho}\sigma'(\overline{\rho})}\overline{\rho}^{-n}=1.337604\cdot(2.584243)^n.$$

A Carlitz composition with zeros allowed can have at most $2n+1$ parts. It is therefore of interest to compare the asymptotic formula for $\overline{c}(n)$ with the total number of compositions (zeros allowed) having at most $2n+1$ parts. This number is given by

$$\sum_{k=1}^{2n+1}\binom{n+k-1}{n}=\binom{3n+1}{n+1}\sim\frac{3\sqrt{3}}{2\sqrt{\pi n}}\Big(\frac{27}{4}\Big)^n.$$

Note that $\frac{27}{4}=6.75$.

6. 1–COMPOSITIONS

If we impose the conditions $a_2 \le a_1 + 1$, $a_3 \le a_2 + 1$, etc. on an ordinary composition, we encounter a different family of restricted compositions which are termed "1–composition" in [2,15].

We want to demonstrate that the technique of adding a new slice applies also very well in this context.

Again, we are using the generating functions $f_k(z,u)$ for 1–compositions enumerated by size and last part. Assume that the last part is j, which is coded by u^j. Then the next part can be any number between 1 and $j+1$. In other words, we must replace u^j by

$$(zu) + (zu)^2 + \cdots + (zu)^{j+1} = \frac{zu(1-(zu)^{j+1})}{1-zu}.$$

In terms of generating functions this substitution translates into

$$f_{k+1}(z,u) = \frac{zu}{1-zu} f_k(z,1) - \frac{z^2u^2}{1-zu} f_k(z,zu)$$

for $k \ge 1$ and $f_1(z,u) = \frac{zu}{1-zu}$. With $F(z,u) = \sum_{k\ge1} f_k(z,u)$ this means

$$F(z,u) = \frac{zu}{1-zu} F(z,1) - \frac{z^2u^2}{1-zu} F(z,zu) + \frac{zu}{1-zu},$$

and upon iteration

$$1 + F(z,1) = \frac{1}{1-\sigma(z)},$$

with

$$\sigma(z) = \sum_{j\ge1} \frac{(-1)^{j-1} z^{j^2}}{(1-z)(1-z^2)\dots(1-z^j)}.$$

There is again a dominant pole at $\rho = 0.576148$, and $\rho\sigma'(\rho) = 1.089257$, so the number of 1–compositions of n is asymptotic to

$$\frac{1}{\rho\sigma'(\rho)} \rho^{-n} = 0.918056 \cdot (1.735662)^n.$$

Now we can also count how many times the condition $a_i \le a_{i+1} + 1$ is *not* satisfied. If we use an auxiliary variable v as we did before, it means that we have to replace u^j by

$$\frac{zu(1-(zu)^{j+1})}{1-zu} + \frac{(zu)^{j+2}v}{1-zu} = \frac{zu(1-(1-v)(zu)^{j+1})}{1-zu}.$$

The appropriate changes are then as follows:

$$F(z,u,v) = \frac{zu}{1-zu} F(z,1,v) - \frac{z^2u^2(1-v)}{1-zu} F(z,zu,v) + \frac{zu}{1-zu},$$

and

$$1 + F(z, 1, v) = \frac{1}{1 - \sigma(z, v)},$$

with

$$\sigma(z, v) = \sum_{j \geq 1} \frac{(v-1)^{j-1} z^{j^2}}{(1-z)(1-z^2)\dots(1-z^j)}.$$

Differentiating $1 + F(z, 1, v)$ w.r.t. v and setting $v = 1$ gives

$$\frac{z^4}{(1+z)(1-2z)^2}.$$

Reading off the coefficient of z^n and dividing by 2^{n-1} we get the average as

$$\frac{3n-8}{36} + O(2^{-n}).$$

$n\backslash k$	0	1	2	3
1	1			
2	2			
3	4			
4	7	1		
5	13	3		
6	23	9		
7	41	23		
8	72	55	1	
9	127	123	6	
10	222	267	23	
11	388	561	75	
12	677	1150	220	1

TABLE 3. Compositions by size and number of adjacent parts a, b with $b > a + 1$

If we consider "0–compositions" ($a_1 \leq a_2 \leq \dots$), which actually means *partitions*, and perform the same analysis, we find the functional equation

$$F(z, u) = \frac{zu}{1-zu} F(z, 1) - \frac{zu}{1-zu} F(z, zu) + \frac{zu}{1-zu},$$

and upon iteration

$$1 + F(z, 1) = \frac{1}{1 - \sigma(z)},$$

with

$$\sigma(z) = \sum_{j \geq 1} \frac{(-1)^{j-1} z^{\binom{j+1}{2}}}{(1-z)(1-z^2)\dots(1-z^j)}.$$

Since

$$1+F(z,1)=\prod_{k\ge 1}\frac{1}{1-z^k},$$

we find in this way one of *Euler's partition identities*, [1].

This time, there is no dominant pole, since the equation $\sigma(z)=1$ has no solution inside the unit circle, and the asymptotics are harder. The interested reader can find the asymptotics (originally by Hardy, Ramanujan, and Rademacher), in the book [1]. Here, one can also find information about the famous Rogers–Ramanujan identities which are close in spirit to "1–compositions." A further paper where such generating functions appear, is [13]; however, we don't intend to be encyclopedic.

7. Concluding Remarks

It should be clear by now that several related quantities can also be treated along the lines of this paper. We mention in particular Carlitz composition with zeros where one can also investigate the parameters that we considered for ordinary Carlitz compositions. In all instances, variances could be computed. 2–compositions etc. could also be done.

A harder problem that we don't know how to do at the moment is the number of different part sizes in Carlitz compositions (for ordinary compositions see [10,12]).

Also, it seems that the rows of all the tables given in this paper are unimodal (apart from the last entry of each row in Table 2). These observations would still require proof.

The referee has kindly informed us about a different approach using *Smirnov words*; see [7; p. 69]. This approach would give some (but not all) of our generating functions, and it is worthwhile to sketch it here.

Think about the parts $1,2,\dots$ of a composition as words and use letters $x_1,x_2,\dots$. Then Smirnov words are those with adjacent letters being different. By replacing each letter x_i by a sequence $x_i/(1-x_i)$ of this letter, all words are obtained. But this is a trivial set (or the generating function is very simple, depending on how things are phrased). With an obvious notation we get

$$S\Big(\frac{x_1}{1-x_1},\frac{x_2}{1-x_2},\dots\Big)=\frac{1}{1-\sum_{i\ge 1}x_i}\,.$$

Since

$$\frac{x}{1-x}=y\qquad\longleftrightarrow\qquad\frac{y}{1+y}=x,$$

the above relation can be *inverted*:

$$S(y_1,y_2,\dots)=\frac{1}{1-\sum_{i\ge 1}\frac{x_i}{1+x_i}}\,.$$

Now for compositions, we should replace each y_i by z^i, which gives the following alternative representation of the generating function $C(z)$;

$$C(z)=\frac{1}{1-\sum_{i\ge 1}\frac{z^i}{1+z^i}}\,.$$

It is not hard to see that the two version coincide; one has to show that

$$\sum_{i\geq 1} \frac{z^i}{1+z^i} = \sum_{j\geq 1} \frac{z^j(-1)^{j-1}}{1-z^j}.$$

For that, one expands the geometric series and interchanges the order of the summations.

(For the other generating functions we would also get versions without the alternating sign $(-1)^i$.)

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