KIRKMAN'S HYPOTHESIS REVISITED

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Abstract. Watson proved Kirkman's hypothesis (partially solved by Cayley). Using Lagrange Inversion, we drastically shorten Watson's computations and generalize his results at the same time.

Kirkman's hypothesis [3] is (in changed notation) the formula

$$\begin{aligned}\sum_{m=0}^{M}\sum_{n=0}^{N}\frac{1}{m+1}\binom{m+n}{n}\binom{2m+n+2}{m+n+2}\ \times \\ \times\ \frac{1}{M-m+1}\binom{M-m+N-n}{N-n}\binom{2(M-m)+N-n+2}{M-m+N-n+2} \\ =\frac{2}{M+2}\binom{M+N+1}{N}\binom{2M+N+4}{M+N+4}.\end{aligned}$$

Kirkman could not prove it, but Cayley [1] proved the special case $N=0$ in 1857. After more than hundred years, Watson [5] proved Kirkman's hypothesis by establishing the following power series expansions. Set

$$\psi(z,w):=\frac{1-w-2z-\sqrt{(1-w)^2-4z}}{2z(z+w)},$$

then

$$\begin{aligned}\psi(z,w)&=\sum_{m,n}\frac{1}{m+1}\binom{m+n}{n}\binom{2m+n+2}{m+n+2}z^mw^n\,,\\ \psi^2(z,w)&=\sum_{m,n}\frac{2}{m+2}\binom{m+n+1}{n}\binom{2m+n+4}{m+n+4}z^mw^n\,.\end{aligned}$$

Of course, Kirkman's hypothesis follows from this by writing $\psi\cdot\psi=\psi^2$ and comparing coefficients. However, Watson's derivation of these two expansions required quite a bit of computation, in particular he treated both cases differently and separately.

Here, we present an extremely simple computation using the *Lagrange inversion formula* that has the advantage of not only treating both cases together but rather finding the power series expansion for $\psi^p(z,w)$ for general p. We refer for the Lagrange inversion formula to [2, 6].

The quadratic equation satisfied by $\psi(z,w)$ is

$$z(z+w)\psi^2(z,w)+(2z+w-1)\psi(z,w)+1=0\,.$$

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Writing $\psi = y/z$ and rearranging leads to the following equation of Lagrange type:

$$y = z\frac{(1+y)^2}{1-w(1+y)} .$$

With the Lagrange inversion formula we obtain:

$$\begin{aligned}
[z^m w^n]\psi^p(z,w) = [z^{m+p}w^n]y^p(z,w) &= \frac{p}{m+p}[y^m w^n]\left(\frac{(1+y)^2}{1-w(1+y)}\right)^{m+p} \\
&= \frac{p}{m+p}\binom{m+n+p-1}{n}[y^m](1+y)^{2m+n+2p} \\
&= \frac{p}{m+p}\binom{m+n+p-1}{n}\binom{2m+n+2p}{m+n+2p} .
\end{aligned}$$

This leads with $\psi^r \cdot \psi^s = \psi^p$ to the convolution formula *(generalized Kirkman hypothesis)*:

$$\begin{aligned}
&\sum_{m=0}^{M}\sum_{n=0}^{N}\frac{r}{m+r}\binom{m+n+r-1}{n}\binom{2m+n+2r}{m+n+2r} \times \\
&\qquad\qquad \times \frac{s}{M-m+s}\binom{M-m+N-n+s-1}{N-n}\binom{2(M-m)+N-n+2s}{M-m+N-n+2s} \\
&= \frac{p}{M+p}\binom{M+N+p-1}{N}\binom{2M+N+2p}{M+N+2p} .
\end{aligned}$$

For other results of Kirkman's, treated with the Lagrange inversion formula, see [4, Ex. 6.33-c].

References

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