Returns to the Origin for Random Walks on $\mathbb{Z}$ Revisited

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Chrysafi and Bradley [1] consider symmetric random walks, defined as follows: Let X_k, $k = 1, 2, \ldots$ be independent and identically distributed random variables with $\mathbb{P}\{X_k = 1\} = \mathbb{P}\{X_k = -1\} = \frac{1}{2}$. Then

$$S_m = \sum_{k=1}^{m} X_k \qquad \text{with} \qquad S_0 = 0$$

is a simple random walk starting at 0. The authors considered only walks of even length $m = 2n$ and were interested in the random variable $R = R_n$, defined to be the NUMBER OF RETURNS TO THE ORIGIN in a walk of length $2n$, i.e., the number of times $S_i = 0$ happens, for $i = 1, \ldots, 2n$. They computed moments up to $\mathbb{E}[R^6]$ and ask for a closed formula for $\mathbb{E}[R^k]$ and also whether $\mathbb{E}[R^k] \sim c_k n^{k/2}$ holds.

The answers to these questions can be found in [4], compare also [5]. There, the *factorial moments* $\mathbb{E}[R^{\underline{k}}]$ where computed. We state the formula only for even n:

$$\mathbb{E}[R_n^{\underline{k}}] = k! \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \binom{\frac{i}{2} + n}{n}.$$

Ordinary moments can be recovered from these formulæ as linear combinations with *Stirling numbers of the second kind (Stirling subset numbers)*, see [3]:

$$\mathbb{E}[R_n^k] = \sum_{i=0}^{k} \left\{ {k \atop i} \right\} \mathbb{E}[R_n^{\underline{i}}].$$

To answer the asymptotics question, we use generating functions:

$$\mathbb{E}[R_n^{\underline{k}}] = 4^{-n}k![z^n]\frac{\left(1-\sqrt{1-4z}\,\right)^k}{(1-4z)^{1+\frac{k}{2}}},$$

and *singularity analysis of generating functions*, as nicely described in [2]. One must expand around the dominant singularity (here $z = \frac{1}{4}$):

$$\frac{\left(1-\sqrt{1-4z}\,\right)^k}{(1-4z)^{1+\frac{k}{2}}} \sim (1-4z)^{-1-\frac{k}{2}}$$

and use a transfer theorem:

$$\mathbb{E}[R_n^{\underline{k}}] \sim 4^{-n}k![z^n](1-4z)^{-1-\frac{k}{2}} = k!\binom{\frac{k}{2}+n}{n} \sim \frac{k!}{(k/2)!}n^{k/2}.$$

For odd $k = 2j+1$, the factor may be rewritten as follows:

$$\frac{k!}{(k/2)!} = \frac{(2j+1)!}{(j+\frac{1}{2})!} = \frac{2^{2j+1}j!}{\sqrt{\pi}}.$$

For ordinary moments, the leading terms in the asymptotic expansion are the same:

$$\mathbb{E}[R_n^k] \sim \frac{k!}{(k/2)!}n^{k/2}.$$

References

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[4] P. Kirschenhofer and H. Prodinger. The higher moments of the number of returns of a simple random walk. *Adv. in Appl. Probab.*, 26(2):561–563, 1994.

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