On a question of Cooper and Kennedy

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In [2], Cooper and Kennedy note the following: If

$$x_n = ax_{n-1} + bx_{n-2} + cx_{n-3} \ , \tag{1}$$

then

$$x_n^2 = Ax_{n-1}^2 + Bx_{n-2}^2 + Cx_{n-3}^2 + Dx_{n-4}^2 + Ex_{n-5}^2 + Fx_{n-6}^2 \ , \tag{2}$$

where the coefficients A, B, C, D, E, F may be expressed in terms of a, b, c. They ask: **Is there a similar formula for third powers?** The answer is: **YES**. The reason is the following: Sequences which are solutions of linear recurrences with constant coefficients have ordinary generating functions which are *rational*. Conversely, if a rational function has no pole in $z = 0$, its Taylor coefficients fulfill a linear recurrences with constant coefficients. If

$$f := \sum_{n\ge0} a_n z^n \qquad \text{and} \qquad g := \sum_{n\ge0} b_n z^n \tag{3}$$

are two (formal) series, their HADAMARD product is defined to be

$$f \odot g := \sum_{n\ge0} a_n b_n z^n \ . \tag{4}$$

And rational functions are closed under the Hadamard product! (See [1, page 85].)

The larger (and maybe even more important) class of *holonomic* functions (solutions of linear differential equations with polynomial coefficients) is also closed under the Hadamard product. Their Taylor coefficients fulfill linear recursions with polynomial coefficients. There is a Maple package, Gfun, which computes (among many other useful things) the Hadamard product. (See [4].)

There is another very useful program, Ekhad, written by Doron Zeilberger ([3]), which should be mentioned. With it, we find for example recursions for the dth powers of the Fibonacci numbers F_n in almost no time. In the following, F_n^d will be a solution of the given recursion.

$d = 1$ $\qquad x_{n+2} - x_{n+1} - x_n = 0$

$d = 2$ $\qquad x_{n+3} - 2x_{n+2} - 2x_{n+1} + x_n = 0$

$d = 3$ $\qquad x_{n+4} - 3x_{n+3} - 6x_{n+2} + 3x_{n+1} + x_n = 0$

$d = 4$ $\qquad x_{n+5} - 5x_{n+4} - 15x_{n+3} + 15x_{n+2} + 5x_{n+1} - x_n = 0$

$d = 5$ $\qquad x_{n+6} - 8x_{n+5} - 40x_{n+4} + 60x_{n+3} + 40x_{n+2} - 8x_{n+1} - x_n = 0$

$d = 6$ $\qquad x_{n+7} - 13x_{n+6} - 104x_{n+5} + 260x_{n+4} + 260x_{n+3} - 104x_{n+2} - 13x_{n+1} + x_n = 0$

References

[1] L. Comtet. *Advanced Combinatorics.* Reidel, Dordrecht, 1974.

[2] C. N. Cooper and R. E. Kennedy. Proof of a result by Jarden by generalizing a proof of Carlitz. *Fibonacci Quarterly,* 33:304–310, 1995.

[3] M. Petkovsek, H. Wilf, and D. Zeilberger. $A = B$. A.K. Peters, Ltd., 1996.

[4] Bruno Salvy and Paul Zimmermann. GFUN: a Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Transactions on Mathematical Software,* 20(2):163–167, 1994.

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