A CURIOUS IDENTITY PROVED BY CAUCHY'S INTEGRAL FORMULA

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Simons [2] has proved the identity

$$\sum_{r=0}^{q} \frac{(-1)^{q+r}(q+r)!(1+x)^r}{(q-r)!r!^2} = \sum_{r=0}^{q} \frac{(q+r)!x^r}{(q-r)!r!^2}; \tag{1}$$

Chapman [1] gave a nice and short proof of it. In this note, I want to give another attractive proof. It uses Cauchy's integral formula to pull out coefficients of generating functions.

We divide (1) by $q!$ and prove the equivalent version

$$S = \sum_{r=0}^{q} \binom{q}{r}\binom{q+r}{r}(-1)^{q+r}(1+x)^r = \sum_{r=0}^{q} \binom{q}{r}\binom{q+r}{r}x^r.$$

We start with the righthand-side:

$$\begin{aligned} S &= [t^q]\sum_{i\ge 0}\binom{q}{i}t^i \cdot \sum_{i\ge 0}\binom{q+i}{i}(tx)^i \\ &= [t^q](1+t)^q \cdot (1-tx)^{-q-1} \\ &= \frac{1}{2\pi i}\oint \frac{dt}{t^{q+1}}(1+t)^q \cdot (1-tx)^{-q-1}. \end{aligned}$$

Now we substitute $t = u/(1-u)$, so that $dt = du/(1-u)^2$ and obtain

$$\begin{aligned} S &= \frac{1}{2\pi i}\oint \frac{du}{(1-u)^2}\frac{(1-u)^{q+1}}{u^{q+1}}(1-u)^{-q} \cdot \left(\frac{1-u(1+x)}{1-u}\right)^{-q-1} \\ &= [u^q]\big(1-u)^q(1-u(1+x)\big)^{-q-1} \\ &= \sum_{r=0}^{q}\binom{-q-1}{r}(-1)^r(1+x)^r\binom{q}{q-r}(-1)^{q-r} \\ &= \sum_{r=0}^{q}\binom{q+r}{r}\binom{q}{r}(1+x)^r(-1)^{q-r}, \end{aligned}$$

which is the lefthand-side.

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References

[1] R. Chapman. A curious identity revisited. *The Mathematical Gazette*, 87:139–141, 2003.
[2] S. Simons. A curious identity. *The Mathematical Gazette*, 85:296–298, 2001.

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