The number of descents in samples of geometric random variables

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ABSTRACT: *For words of length n, generated by independent geometric random variables, we consider the mean and variance, and thereafter the distribution of the number of descents in the words. The cases of strict and weak descents are both considered. We also study the position and height of the first descent as well as the size of the greatest descent over all words of length n.*

1 Introduction

Let X denote a geometrically distributed random variable, i. e. $\mathbb{P}\{X = k\} = pq^{k-1}$ for $k \in \mathbb{N}$ and $q = 1 - p$. The combinatorics of n geometrically distributed independent random variables $X_1, \ldots, X_n$ has attracted recent interest, especially because of applications in computer science. We mention just two areas, the skip list [1, 13, 16, 8] and probabilistic counting [3, 6, 7, 9].

One of the first combinatorial questions investigated for words $a_1 \ldots a_n$, with the letters a_i independently generated according to the geometric distribution, was the number of left-to-right maxima in [14] . In [10] the study of left-to-right maxima was continued, but now the parameters studied were the mean value and mean position of the r-th maximum.

In this article we study descents in a string of n geometrically distributed independent random letters. For example in $w = 221114431$ we have 3 strict and 7 weak descents (equality is included). In the sequel we denote by $D_n(w)$ the number of descents (strict or weak as indicated) in the word w, where w is of length n.

In section 2 we study the mean and variance of $D_n(w)$. Thereafter, in section 3 we study the distribution of the number of descents, which turns out to be Gaussian. Subsequently, in sections 4 and 5 we study the average position and height of the first descent. Finally, in section 6 we determine the expected value of the size of the greatest descent in a word of length n. By reversing the strings we see that exactly the same results hold for *ascents* as for descents in such words. Our treatment to some extent parallels the study of runs in geometric random variables by Grabner and the present authors in [4].

We make use of the following abbreviations: $p = 1 - q$, $Q = 1/q$, $L = \log Q$. Words have length n.

2 The number of descents

As always in the setting of geometrically distributed random variables, we have two choices: to use the '$>$' or the '$\geq$' sign. Let us do the instance of strict descents first.

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2.1 Strict descents

In order to determine the mean and variance of the number of strict descents we will make use of the following decomposition of the set of all words. Here $\{\geq \mathbf{k}\}$ denotes the set $\{k, k+1, \ldots\}$; for a given set A we denote

$$A^+ = \bigcup_{k=1}^{\infty} A^k, \quad A^* = \varepsilon \cup A^+,$$

where ε stands for the empty word. We decompose the set of all words according to runs of 1's, separated by words consisting of larger digits only:

$$\{\geq \mathbf{1}\}^* = \mathbf{1}^*\big(\{\geq \mathbf{2}\}^+\mathbf{1}^+\big)^*\{\geq \mathbf{2}\}^+(\varepsilon + \mathbf{1}^+) + \mathbf{1}^*. \tag{1}$$

We consider a probability generating function $F(z,u)$, where z labels the number of random variables (or length of the word, if you wish), and u counts the number of descents. We should always have $F(z,1) = \frac{1}{1-z}$, which we might use as a check, and a replacement of z by qz, if we increase all letters by 1.

$$F(z,u) = \frac{1}{1-pz}\frac{1}{1-[F(qz,u)-1]\dfrac{pzu}{1-pz}}[F(qz,u)-1]\Big(\frac{pzu}{1-pz}+1\Big)+\frac{1}{1-pz}. \tag{2}$$

Now we differentiate it w.r.t. u, plug in $u=1$, set $G(z) = \frac{\partial}{\partial u}F(z,1)$, and get

$$G(z) = G(qz)\frac{(1-qz)^2}{(1-z)^2} + \frac{pqz^2}{(1-z)^2}.$$

We set $H(z) = (1-z)^2G(z)$ and get $H(z) = H(qz) + pqz^2$. Comparing coefficients, we see that

$$[z^2]H(z) = \frac{pq}{1-q^2} = \frac{q}{1+q},$$

and that the other coefficients are zero. Consequently,

$$H(z) = \frac{q}{1+q}z^2 \quad \text{and} \quad G(z) = \frac{q}{1+q}\frac{z^2}{(1-z)^2}.$$

Theorem 1 *The expected value of the number of descents is given for $n \geq 1$ by*

$$[z^n]G(z) = (n-1)\ \frac{q}{1+q}. \tag{3}$$

Now that we see such a simple result, we will be tempted to look for a simple proof as well. Indeed, since the expectation is additive, we will have $n-1$ times the expectation between two adjacent random variables. This quantity is

$$\sum_{i>j\geq 1} pq^{i-1}\,pq^{j-1} = \frac{q}{1+q}.$$

If we want to compute the variance, such a simple argument seems to be out of reach. Henceforth, we differentiate (2) twice, and use the notation $V(z) = \frac{\partial^2}{\partial u^2}F(z,1)$.

We see

$$V(z) = V(qz)\frac{(1-qz)^2}{(1-z)^2} + \frac{2q^2(1-q)z^3\big(q(1+q)+(1-q^2)z-q^2z^2\big)}{(1+q)^2(1-z)^3(1-qz)},$$

or, with $W(z) = (1-z)^2V(z)$,

$$W(z) = W(qz) - \frac{2(1-q)q^2z(1+(1+q)z+z^2)}{(1+q)^2} + \frac{2q^2}{(1+q)^2}\Big[\frac{1}{1-z} - \frac{1}{1-qz}\Big].$$

From this we see that (with $w_n := [z^n]W(z)$) for $n \geq 4$,

$$w_n = q^n w_n + \frac{2q^2}{(1+q)^2}\big[1-q^n\big], \quad \text{or} \quad w_n = \frac{2q^2}{(1+q)^2}.$$

Furthermore we see that $w_3 = \dfrac{2q^3(1-q)}{(1+q)(1-q^3)}$ and $w_0 = w_1 = w_2 = 0$. Hence

$$V(z) = \frac{2q^3(1-q)}{(1+q)(1-q^3)}\frac{z^3}{(1-z)^2} + \frac{2q^2}{(1+q)^2}\frac{z^4}{(1-z)^3}$$

and

$$[z^n]V(z) = \frac{q^2}{(1+q)^2}(n-2)(n-3) + \frac{2q^3(1-q)}{(1+q)(1-q^3)}(n-2).$$

Adding the expectation and subtracting the square of the expectation, we obtain the variance.

Theorem 2 *The variance of the number of strict descents is given for $n \geq 1$ by*

$$\frac{q(1-q)(1+q^3)}{(1+q)^3(1-q^3)}n - \frac{q(1-3q+q^2)}{(1+q)^2(1+q+q^2)}. \tag{4}$$

2.2 Weak descents

We will use the same decomposition (1), but with a slightly different translation into generating functions.

Without explicitly mentioning it, we will use the same functions as before, adjusted to the new meaning.

$$F(z,u) = \Big(1+\frac{pz}{1-pzu}\Big)\frac{1}{1-[F(qz,u)-1]\dfrac{pzu}{1-pzu}}[F(qz,u)-1]\Big(\frac{pzu}{1-pzu}+1\Big)$$
$$+1+\frac{pz}{1-pzu}. \tag{5}$$

Therefore

$$G(z) = G(qz)\frac{(1-qz)^2}{(1-z)^2} + \frac{pz^2}{(1-z)^2}$$

and $H(z) = H(qz) + pz^2$. Consequently,

$$H(z) = \frac{1}{1+q}z^2 \quad \text{and} \quad G(z) = \frac{1}{1+q}\frac{z^2}{(1-z)^2}.$$

Theorem 3 *The expected value of the number of weak descents is given for $n \ge 1$ by*

$$[z^n]G(z) = (n-1)\,\frac{1}{1+q}. \tag{6}$$

A simple explanation is also available as in the other instance, and therefore omitted.— Now we turn to the variance. We see

$$V(z) = V(qz)\frac{(1-qz)^2}{(1-z)^2} + \frac{2(1-q)z^3\big(1+q+(1-q^2)qz-q^3z^2\big)}{(1+q)^2(1-z)^3(1-qz)},$$

or, with $W(z) = (1-z)^2V(z)$,

$$W(z) = W(qz) - \frac{2(1-q)z(1+(1+q)z+q^2z^2)}{(1+q)^2} + \frac{2}{(1+q)^2}\Big[\frac{1}{1-z} - \frac{1}{1-qz}\Big].$$

From this we see that (with $w_n := [z^n]W(z)$) for $n \ge 4$,

$$w_n = q^n w_n + \frac{2}{(1+q)^2}\big[1-q^n\big], \quad \text{or} \quad w_n = \frac{2}{(1+q)^2}.$$

Furthermore we see that $w_3 = \dfrac{2(1-q)}{(1+q)(1-q^3)}$ and $w_0 = w_1 = w_2 = 0$. Hence

$$V(z) = \frac{2(1-q)}{(1+q)(1-q^3)}\frac{z^3}{(1-z)^2} + \frac{2}{(1+q)^2}\frac{z^4}{(1-z)^3}$$

and

$$[z^n]V(z) = \frac{1}{(1+q)^2}(n-2)(n-3) + \frac{2(1-q)}{(1+q)(1-q^3)}(n-2).$$

Adding the expectation and subtracting the square of the expectation, we obtain the variance.

Theorem 4 *The variance of the number of weak descents is given for $n \ge 1$ by*

$$\frac{q(1-q)(1+q^3)}{(1+q)^3(1-q^3)}n - \frac{q(1-3q+q^2)}{(1+q)^2(1+q+q^2)}. \tag{7}$$

We note that this quantity is the same as in the instance of the strict descents.

3 Distribution of the number of descents

In this section we prove a central limit theorem for the distribution of the number of strict descents. In order to do this, we have to extract further information from the functional equation (2). We observe that the terms on the right-hand side are all simple rational functions, except for the terms containing $F(qz,u)$. From the definition of $F(z,u)$ it is clear that $F(z,u)$ can be written as

$$F(z,u) - 1 = \sum_{n\ge1} z^n f_n(u)$$

for polynomials $f_n(u)$ with $\deg f_n = n$, whose coefficients are positive and ≤ 1. Therefore we have

$$|f_n(u)| \le \frac{|u|^{n+1}-1}{|u|-1}, \quad \text{for } |u|>1. \tag{8}$$

Using $q<1$ and (8) we obtain that $F(qz,u)$ is holomorphic in $|z| < \frac{1}{\sqrt{q}}$, $|u| < \frac{1}{\sqrt{q}}$. Since for $u=z=1$ we have

$$1-\big(F(q,1)-1\big)\frac{p}{1-p} = 0, \quad \text{and} \quad \frac{\partial}{\partial z}\Big(1-(F(qz,u)-1)\frac{pzu}{1-pz}\Big)\Big|_{z=1,u=1} \neq 0$$

there exists a function $f(u)$ holomorphic in a neighbourhood of $u=1$ such that $z=f(u)^{-1}$ solves

$$1-\big(F(qz,u)-1\big)\frac{pzu}{1-pz} = 0$$

and satisfies $f(1)=1$. Furthermore, $|f(e^{it})|<1$ for $0<|t|<\varepsilon$ for some $\varepsilon>0$ by an application of Rouché's theorem. Thus we can write

$$F(z,u)-1 = \frac{g(z,u)}{1-f(u)z} + R(z,u), \tag{9}$$

where $g(z,u)$ and $R(z,u)$ are holomorphic in $|z|<1+\delta$, $|u-1|<\delta$ for some $\delta>0$. Now we are in the general framework of Hwang's quasi-power theorem (cf. [5]) and can deduce the following theorem.

Theorem 5 *The number of strict descents in words of length n produced by independent geometric random variables obeys a central limit law, more precisely*

$$\mathbb{P}\left(D_n(w) \le \frac{q}{1+q}n + t\sqrt{\frac{q(1-q)(1+q^3)}{(1+q)^3(1-q^3)}}\sqrt{n}\right) = \Phi(t) + \mathcal{O}(n^{-\frac{1}{2}}). \tag{10}$$

Similarly we can show

Theorem 6 *The number of weak descents in words of length n produced by independent geometric random variables obeys a central limit law, more precisely*

$$\mathbb{P}\left(D_n(w) \le \frac{1}{1+q}n + t\sqrt{\frac{q(1-q)(1+q^3)}{(1+q)^3(1-q^3)}}\sqrt{n}\right) = \Phi(t) + \mathcal{O}(n^{-\frac{1}{2}}). \tag{11}$$

Remarks In [12] Louchard and Prodinger find an explicit bivariate generating function for samples of geometric random variables with k strictly ascending runs. Since the number of weak descents is one less than the number of strictly ascending runs, we may deduce from their result that in the case of weak descents

$$F(z,u) = \frac{(1-u)P}{1-uP}, \quad \text{where} \quad P = \prod_{n\ge 0}(1+pz(1-u)q^n). \tag{12}$$

To verify this directly one must show that (12) is the solution of the functional equation (5) that satisfies $F(0,u)=1$. This follows easily by substituting (12) in (5) using $F(qz,u) = \frac{(1-u)P}{1+pz(1-u)-uP}$.

Similarly, in the case of strict descents one easily verifies that the solution of the functional equation (2) that satisfies $F(0,u)=1$ is

$$F(z,u)=\frac{1-u}{P-u}, \quad \text{where now} \quad P=\prod_{n\geq 0}(1-pz(1-u)q^n). \tag{13}$$

In both (12) and (13) we may let $q\to 1$ and obtain as expected the bivariate exponential generating function for the Eulerian numbers

$$A(z,u)=\frac{1-u}{e^{(u-1)z}-u}.$$

4 The position of the first descent

4.1 Strict descents

We use the following variation of the number of descents decomposition for the set of all words:

$$\{\geq \mathbf{1}\}^* = \mathbf{1}^*\{\geq \mathbf{2}\}^+\big(\mathbf{1}^+\{\geq \mathbf{2}\}^+\big)^*(\varepsilon+\mathbf{1}^+)+\mathbf{1}^*. \tag{14}$$

We consider a probability generating function $F(z,u)$, where z labels the number of random variables, and u marks the initial position of the first descent. In working with this decomposition it is convenient to adopt the convention that a nondecreasing word has a descent at its end.

$$F(z,u)=\frac{1}{1-pzu}[F(qz,u)-1]\frac{1}{1-[F(qz,1)-1]\dfrac{pz}{1-pz}}\frac{1}{1-pz}+\frac{1}{1-pzu}, \tag{15}$$

which simplifies to

$$F(z,u)=\frac{1}{1-pzu}[F(qz,u)-1]\frac{1-qz}{1-z}+\frac{1}{1-pzu}. \tag{16}$$

Now we differentiate it w.r.t. u, plug in $u=1$, set $G(z)=\frac{\partial}{\partial u}F(z,1)$, and get

$$G(z)=\frac{1}{1-z}G(qz)\frac{1-qz}{1-pz}+\frac{pz}{1-pz}.$$

We set $H(z)=(1-z)G(z)$ and get $(1-pz)H(z)=H(qz)+pz$. Comparing coefficients, we see that $h_n:=[z^n]H(z)$ satisfies $h_1=1$ and for $n\geq 2$, $h_n-ph_{n-1}=q^nh_n$. By iterating this recurrence, $h_n=p^n/(q;q)_n$, with the notation $(a;q)_n=(1-a)(1-aq)\dots(1-aq^{n-1})$. Consequently,

$$H(z)=\sum_{n\geq 1}\frac{p^nz^n}{(q;q)_n} \quad \text{and} \quad G(z)=\frac{1}{1-z}\sum_{n\geq 1}\frac{p^nz^n}{(q;q)_n}=\frac{1}{1-z}\prod_{i\geq 0}\frac{1}{1-pzq^i},$$

by one of Euler's partition identities.

Our convention gives a contribution of n for the position of the first descent in each nondecreasing word, whereas 0 might be more natural. If we prefer the latter convention then we should subtract n times the probability $p^n/(q;q)_n$ of a nondecreasing word from $[z^n]G(z)$ to compute the average.

Theorem 7 *The average position of the first descent is given for $n \geq 1$ by*

$$\sum_{j=1}^{n} \frac{p^j}{(q;q)_j} - \frac{p^n}{(q;q)_n} n = \sum_{j=1}^{n} \frac{1}{[j]_q!} - \frac{n}{[n]_q!}. \tag{17}$$

(We used the notation $[n]_q! = (q;q)_n/p^n$.) Note that as $q \to 1$ we find that the average position of the first descent in a permutation equals $\sum_{j=1}^{n} \frac{1}{j!} - \frac{n}{n!} \to e-1$, as $n \to \infty$, in agreement with results of Knuth [11]. As $q \to 0$, we obtain a word of all **1**'s and hence 0 for the average position of the first descent as expected.

Previously, using a different approach, Prodinger [15] studied the length of the r-th ascending run in a geometrically distributed word. The length of the first weakly (strictly) ascending run corresponds to the position of the first strict (weak) descent, respectively.

4.2 Weak descents

We use the same decomposition (14) for the set of all words.

We consider a probability generating function $F(z,u)$, where z labels the number of random variables, and u marks the initial position of the first weak descent. Again it is convenient to adopt a convention that a strictly increasing word has a weak descent at its end. We split up our decomposition into the cases of zero or one **1**, or two or more **1**'s. In the latter case a weak descent has already occurred with initial position equal to 1.

$$\begin{aligned} F(z,u) &= (1+pzu)\frac{[F(qz,u)-1]}{1-[F(qz,1)-1]\dfrac{pz}{1-pz}}\frac{1}{1-pz} \\ &\quad + \frac{p^2z^2u}{1-pz}\frac{[F(qz,1)-1]}{1-[F(qz,1)-1]\dfrac{pz}{1-pz}}\frac{1}{1-pz} + \Big(1+\frac{pzu}{1-pz}\Big), \end{aligned} \tag{18}$$

which simplifies to

$$F(z,u) = (1+pzu)[F(qz,u)-1]\frac{1-qz}{1-z} + \frac{p^2z^2u}{1-pz}\frac{qz}{1-z} + 1 + \frac{pzu}{1-pz}. \tag{19}$$

Now we differentiate it w.r.t. u, plug in $u=1$, set $G(z) = \frac{\partial}{\partial u}F(z,1)$, and get

$$G(z) = \frac{1}{1-z}\Big(G(qz)(1-qz)(1+pz) + pz\Big).$$

We set $H(z) = (1-z)G(z)$ and get $H(z) = H(qz)(1+pz) + pz$. By comparing coefficients we find that

$$H(z) = \sum_{n\geq 1} \frac{p^n q^{\binom{n}{2}} z^n}{(q;q)_n} \quad \text{and} \quad G(z) = \frac{1}{1-z}\sum_{n\geq 1}\frac{p^n q^{\binom{n}{2}} z^n}{(q;q)_n} = \frac{1}{1-z}\prod_{i\geq 0}(1+pzq^i),$$

by another of Euler's partition identities.

Since our convention gives a contribution of n rather than 0 for the position of the first descent in each increasing word, we subtract n times the probability $p^n q^{\binom{n}{2}}/(q;q)_n$ of a strictly increasing word from $[z^n]G(z)$ to compute the average.

Theorem 8 *The average position of the first descent is given for $n \geq 1$ by*

$$\sum_{j=1}^{n} \frac{p^j q^{\binom{j}{2}}}{(q;q)_j} - \frac{p^n q^{\binom{n}{2}}}{(q;q)_n} n = \sum_{j=1}^{n} \frac{q^{\binom{j}{2}}}{[j]_q!} - \frac{n q^{\binom{n}{2}}}{[n]_q!}. \tag{20}$$

Note that as $q \to 1$ we find again that the average position of the first descent in a permutation equals $\sum_{j=1}^{n} \frac{1}{j!} - \frac{n}{n!} \to e - 1$, as $n \to \infty$. As $q \to 0$, we obtain a word of all **1**'s and hence 1 for the average position of the first weak descent as expected.

5 The height of the first descent

5.1 Strict descents

We stay with decomposition (14) and consider a probability generating function $F(z,u)$, where now u marks the initial height of the first descent. Once again we adopt the convention that a nondecreasing word has a descent at its end.

$$F(z,u) = \frac{u}{1-pz}[F(qz,u)-1]\frac{1}{1-[F(qz,1)-1]\dfrac{pz}{1-pz}}\frac{1}{1-pz} + \Big(1 + \frac{pzu}{1-pz}\Big), \tag{21}$$

which simplifies to

$$F(z,u) = \frac{u}{1-pz}[F(qz,u)-1]\frac{1-qz}{1-z} + 1 + \frac{pzu}{1-pz}. \tag{22}$$

Now we differentiate it w.r.t. u, plug in $u = 1$, set $G(z) = \frac{\partial}{\partial u}F(z,1)$, and get

$$G(z) = \frac{1}{1-z}G(qz)\frac{1-qz}{1-pz} + \frac{z}{1-z}.$$

We set $H(z) = (1-z)G(z)$ and get $(1-pz)H(z) = H(qz) + z(1-pz)$. This leads to

$$H(z) = z + \frac{q}{p}\sum_{n\geq 1}\frac{p^n z^n}{(q;q)_n} \quad \text{and} \quad G(z) = \frac{z}{1-z} + \frac{q}{p}\frac{1}{1-z}\sum_{n\geq 1}\frac{p^n z^n}{(q;q)_n}.$$

Theorem 9 *The average initial height of the first descent is given for $n \geq 1$ by*

$$\frac{q}{p}\sum_{j=1}^{n}\frac{p^j}{(q;q)_j} + 1 = \frac{q}{p}\sum_{j=1}^{n}\frac{1}{[j]_q!} + 1. \tag{23}$$

As $q \to 1$ the sum tends to infinity indicating that the average height of the first descent in a permutation grows as a function of the size of the permutation. As $q \to 0$, we obtain a word of all **1**'s and hence 1 for the average height of the first strict descent as expected from our convention that such a word has a strict decent at the end.

5.2 Weak descents

Again we use decomposition (14) and consider a probability generating function $F(z,u)$, where now u marks the initial height of the first weak descent. Once again we adopt the convention that a strictly increasing word has a descent at its end.

We split up our decomposition into the cases of zero or one $\mathbf{1}$, or two or more $\mathbf{1}$'s. In the latter case a weak descent has already occurred with initial height equal to 1:

$$\begin{aligned} F(z,u) &= u(1+pz)\frac{[F(qz,u)-1]}{1-[F(qz,1)-1]\dfrac{pz}{1-pz}}\frac{1}{1-pz} \\ &\quad + \frac{p^2z^2u}{1-pz}\frac{[F(qz,1)-1]}{1-[F(qz,1)-1]\dfrac{pz}{1-pz}}\frac{1}{1-pz} + 1 + \frac{pzu}{1-pz}. \end{aligned} \tag{24}$$

Again we simplify, differentiate w.r.t. u, plug in $u=1$, set $G(z) = \frac{\partial}{\partial u}F(z,1)$, and get

$$G(z) = \frac{1}{1-z}\Big(G(qz)(1-qz)(1+pz)+z\Big).$$

We set $H(z) = (1-z)G(z)$ and get $H(z) = H(qz)(1+pz)+z$. This leads to

$$H(z) = \frac{1}{p}\sum_{n\ge1}\frac{p^n q^{\binom{n}{2}}z^n}{(q;q)_n} \quad \text{and} \quad G(z) = \frac{1}{p}\frac{1}{1-z}\sum_{n\ge1}\frac{p^n q^{\binom{n}{2}}z^n}{(q;q)_n}.$$

Theorem 10 *The average initial height of the first weak descent is given for $n \ge 1$ by*

$$\frac{1}{p}\sum_{j=1}^{n}\frac{p^j q^{\binom{j}{2}}}{(q;q)_j} = \frac{1}{p}\sum_{j=1}^{n}\frac{q^{\binom{j}{2}}}{[j]_q!}. \tag{25}$$

As $q \to 1$ we again find that the average height of the first descent in a permutation grows with n. As $q \to 0$, we obtain a word of all $\mathbf{1}$'s and hence obtain 1 for the average height of the first weak descent as expected.

6 The maximum size of a descent

We study the following parameter X:

$$X := \max\{x_i - x_{i+1} \mid 1 \le i < n \ \&\ x_i \ge x_{i+1}\}; \tag{26}$$

it is consistent to assign the value 0 if no pair of consecutive letters satisfies the condition. It is more convenient to think from right to left, and we may still call it X; the statistics are the same:

$$X := \max\{x_{i+1} - x_i \mid 1 \le i < n \ \&\ x_i \le x_{i+1}\}.$$

Fix a parameter h and consider all words with $X \le h$. That is, that for all $1 \le i < n$, we must have that $x_{i+1} \in \{1, 2, \dots, x_i + h\}$.

The adding-a-new-slice technique works here: Let $f_n(u)$ be such that $[u^i]f_n(u)$ is the probability that a word of length n has $X \le h$ and last letter equal to i. Then $f_1(u) = \sum_{i\ge1} pq^{i-1}u^i = \frac{pu}{1-qu}$, and we have the replacement rule

$$u^i \longrightarrow pu + pqu^2 + \cdots + pq^{i+h-1}u^{i+h} = pu\frac{1-(qu)^{i+h}}{1-qu}.$$

Consequently,

$$f_{n+1}(u) = \frac{pu}{1-qu}f_n(1) - \frac{pq^h u^{h+1}}{1-qu}f_n(qu). \tag{27}$$

Now define $F(u) = F(z,u) = \sum\limits_{n\ge1} z^n f_n(u)$. Then, upon summing,

$$\sum_{n\ge1} z^{n+1} f_{n+1}(u) = \frac{pu}{1-qu}\sum_{n\ge1} z^{n+1} f_n(1) - \frac{pq^h u^{h+1}}{1-qu}\sum_{n\ge1} z^{n+1} f_n(qu)$$

or

$$F(z,u) - zf_1(u) = \frac{puz}{1-qu}F(z,1) - \frac{pq^h u^{h+1} z}{1-qu}F(z,qu).$$

Now we iterate the functional equation

$$F(u) = \frac{puz}{1-qu} + \frac{puz}{1-qu}F(1) - \frac{pq^h u^{h+1} z}{1-qu}F(qu). \tag{28}$$

Setting $u = 1$ and collecting terms:

$$F(1) = \sum_{k\ge1} \frac{(-1)^{k-1}p^k q^{\binom{k}{2}(h+1)} z^k}{(q)_k} + F(1)\sum_{k\ge1} \frac{(-1)^{k-1}p^k q^{\binom{k}{2}(h+1)} z^k}{(q)_k}.$$

It is better now to include the empty word and thus work with

$$1 + F(1) = \left(1 - \sum_{k\ge1} \frac{(-1)^{k-1}p^k q^{\binom{k}{2}(h+1)} z^k}{(q)_k}\right)^{-1}. \tag{29}$$

We must study the dominant zeros ρ_h of

$$1 - \sum_{k\ge1} \frac{(-1)^{k-1}p^k q^{\binom{k}{2}(h+1)} z^k}{(q)_k} = 1 - \sigma_h(z) = 0. \tag{30}$$

Bootstrapping is the appropriate technique in this instance:

$$1 - z + \frac{q^{h+1}z^2}{1+q} \sim 0.$$

Set $\rho_h = 1 + \varepsilon_h$, then $\varepsilon_h \sim \frac{q^{h+1}}{1+q}$. We have $1 + F(1) \sim \dfrac{A_h}{1 - z/\rho_h}$ with

$$A_h = \frac{1}{\rho_h \sigma'(\rho_h)} \sim 1\Big/\sum_{k\ge1} \frac{(-1)^{k-1}kp^k q^{\binom{k}{2}(h+1)}}{(q)_k}. \tag{31}$$

So the probability that $X \le h$ is approximated by

$$A_h \rho^{-n} \sim A_h \Big(1 - \frac{q^{h+1}}{1+q}\Big)^n \sim A_h \exp\Big(-\frac{q^{h+1} n}{1+q}\Big).$$

As one can check, $A_h \sim 1$, and thus we may work with $\exp(-\frac{q^{h+1}n}{1+q})$.

We now apply the Mellin transform (cf. [2]) to the function $f(t) = \sum_{h\ge 0} \big(1 - \exp(-t\frac{q^{h+1}}{1+q})\big)$. This yields the transformed function

$$f^*(s) = \Gamma(s) \frac{(1+q)^s}{1-q^s}, \quad \text{for } -1 < \Re s < 0.$$

Application of the Mellin inversion formula, shifting the line of integration to the right and collecting (negative) residues yields

$$\begin{aligned} f(t) &= \frac{1}{2\pi i} \int_{-\frac12 - i\infty}^{-\frac12 + i\infty} \Gamma(s) \frac{(1+q)^s}{1-q^s} t^{-s}\, ds \\ &= -\sum_{k\in\mathbb{Z}} \operatorname{Res} f^*(s) t^{-s}\big|_{s=\frac{2k\pi i}{L}} - \frac{1}{2\pi i} \int_{\frac12 - i\infty}^{\frac12 + i\infty} \Gamma(s) \frac{(1+q)^s}{1-q^s} t^{-s}\, ds. \end{aligned} \tag{32}$$

The residues are easily computed to be

$$-\operatorname{Res} \Gamma(s) \frac{(1+q)^s}{1-q^s} t^{-s} \Big|_{s=0} = \log_Q t + \frac{\gamma}{L} - \frac{\log(Q+1)}{L} + \frac12, \tag{33}$$

$$-\operatorname{Res} \Gamma(s) \frac{(1+q)^s}{1-q^s} t^{-s} \Big|_{s=\frac{2k\pi i}{L}} = -\frac{1}{L} \Gamma\Big(\frac{2k\pi i}{L}\Big) \Big(\frac{t}{Q+1}\Big)^{-\frac{2k\pi i}{L}}. \tag{34}$$

This yields

Theorem 11 *The mean value of the size of the greatest descent $M_n(w)$ in a string of n geometric random variables satisfies*

$$\mathbb{E} M_n(w) \sim \log_Q n + \frac{\gamma}{L} - \frac{\log(Q+1)}{L} + \frac12 - \delta\Big(\log_Q \Big(\frac{n}{Q+1}\Big)\Big), \tag{35}$$

where $\delta(x)$ is a periodic function of mean zero, period one and small magnitude, with Fourier series

$$\delta(x) = \frac{1}{L} \sum_{k\ne 0} \Gamma\Big(-\frac{2k\pi i}{L}\Big) e^{2k\pi i x}.$$

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