A q–ANALOGUE OF A FORMULA OF HERNANDEZ OBTAINED BY INVERTING A RESULT OF DILCHER

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ABSTRACT. We prove a q–analogue of the formula

$$\sum_{1\le k\le n}\binom{n}{k}(-1)^{k-1}\sum_{1\le i_1\le i_2\le\cdots\le i_m=k}\frac{1}{i_1i_2\dots i_m}=\sum_{1\le k\le n}\frac{1}{k^m}$$

by inverting a formula due to Dilcher.

1. THE IDENTITIES

Hernández in [5] proved the following identity:

$$\sum_{1\le k\le n}\binom{n}{k}(-1)^{k-1}\sum_{1\le i_1\le i_2\le\cdots\le i_m=k}\frac{1}{i_1i_2\dots i_m}=\sum_{1\le k\le n}\frac{1}{k^m}. \tag{1}$$

However this identity does not really require a proof, since we will show that it is just an inverted form of an identity of Dilcher; [3]

$$\sum_{1\le k\le n}\binom{n}{k}(-1)^{k-1}\frac{1}{k^m}=\sum_{1\le i_1\le i_2\le\cdots\le i_m\le n}\frac{1}{i_1i_2\dots i_m}. \tag{2}$$

Define for $k\ge 1$

$$a_k:=-\sum_{1\le i_1\le i_2\le\cdots\le i_m=k}\frac{1}{i_1i_2\dots i_m}\qquad\text{and}\qquad b_k:=\frac{1}{k^m},$$

and $a_0=b_0=0$, then the identities are

$$\sum_{0\le k\le n}\binom{n}{k}(-1)^k a_k=\sum_{0\le k\le n}b_k,$$
$$\sum_{0\le k\le n}\binom{n}{k}(-1)^k b_k=\sum_{0\le k\le n}a_k.$$

They are *inverse relations*, as can be seen by introducing ordinary generating functions $A(z)=\sum a_nz^n$ and $B(z)=\sum b_nz^n$. Then they are

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$$\frac{1}{1-z}A\Big(\frac{z}{z-1}\Big)=\frac{1}{1-z}B(z),$$
$$\frac{1}{1-z}B\Big(\frac{z}{z-1}\Big)=\frac{1}{1-z}A(z).$$

However

$$w=\frac{z}{z-1}\longleftrightarrow z=\frac{w}{w-1},$$

and the proof is finished

We note that Dilcher's sum appears also in disguised form in [4].

2. A q–analogue

Dilcher's formula (2) was only a corollary of his elegant q–version;

$$\sum_{1\le k\le n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^{k-1}\frac{q^{\binom{k+1}{2}+(m-1)k}}{(1-q^k)^m}=\sum_{1\le i_1\le i_2\le\dots\le i_m\le n}\frac{q^{i_1}}{1-q^{i_1}}\cdots\frac{q^{i_m}}{1-q^{i_m}}.$$

Here, $\begin{bmatrix}n\\k\end{bmatrix}_q$ denotes the Gaussian polynomial

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

with

$$(x;q)_n:=(1-x)(1-xq)\dots(1-xq^{n-1}).$$

Apart from Dilcher's paper [3], the article [1] is also of some relevance in this context.

Therefore it is a natural question to find a q–analogue of Hernández' formula, or, what amount to the same, to find the appropriate inverse relations for the q–analogues.

We state them in the following lemma which is almost surely not new. However, I found it more appealing to derive it myself rather than hunting for it in the vast q–literature.

Lemma 1.

$$\sum_{k=0}^n b_k=\sum_{k=0}^n\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^kq^{\binom{k}{2}}a_k, \tag{3}$$
$$\sum_{k=0}^n q^{-k}a_k=\sum_{k=0}^n\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^kq^{-kn+\binom{k}{2}}b_k. \tag{4}$$

Proof. First note that it is much easier to prove this than to find it (it took me a few hours, but I am not too experienced).

We note that, as is usual in such cases, it is sufficient to prove it for a basis of the vector space of polynomials. Here, we choose $a_n=q^nx^n(1-\frac{1}{x})$ for $n\ge1$ and $a_0=1$.

We need the following standard formulæ that are consequences of the q–binomial theorem (see e. g. [2]):

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k x^k = (x;q)_n,$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (x;q)_k x^{n-k} = 1$$

We plug the form of a_n into the right hand side of (3) and obtain

$$(1-\tfrac{1}{x}) \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^k x^k = (1-\tfrac{1}{x})(qx;q)_n = -\tfrac{1}{x}(x;q)_{n+1} = \sum_{k=0}^{n} b_k.$$

Thus

$$b_n = -\tfrac{1}{x}(x;q)_{n+1} + \tfrac{1}{x}(x;q)_n = -\tfrac{1}{x}(x;q)_n(1-xq^n-1) = q^n(x;q)_n.$$

We are done if these values of a_n and b_n also satisfy the relation (4). Note that (4) can be rewritten as

$$\sum_{k=0}^{n} q^{-k} a_k = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{1/q} (-1)^k q^{-\binom{k+1}{2}} b_k.$$

We plug b_n into the right hand side of (4) and obtain

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{1/q} (-1)^k q^{-\binom{k+1}{2}} q^k (x;q)_k = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{1/q} (-1)^k x^k (\tfrac{1}{x}; \tfrac{1}{q})_k = x^n = \sum_{k=0}^{n} q^{-k} a_k.$$

Thus

$$q^{-n} a_n = x^n - x^{n-1} = x^n(1-\tfrac{1}{x}).$$

We would like to remark that an alternative proof could be found by dealing with the matrices of *connecting coefficients*.

Define matrices

$$T := \left[\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} \right]_{n,k}, \qquad S := \left[\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-kn+\binom{k}{2}} \right]_{n,k},$$

$$U = [\mathbf{1}_{n\geq k}]_{n,k}, \quad \text{and} \quad V = [q^{-k}\mathbf{1}_{n\geq k}]_{n,k}.$$

Then we have to prove that $S = VT^{-1}U$.

This is not too hard, since

$$T^{-1} = \left[\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-kn+\binom{k+1}{2}} \right]_{n,k},$$

and

$$T^{-1}U = \left[\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^k q^{-n(k-1)+\binom{k}{2}} \right]_{n,k}.$$

Theorem 1. [q–analogue of Hernández' formula]

$$\sum_{1\le k\le n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{k-1} q^{-kn+\binom{k}{2}} \sum_{1\le i_1\le i_2\le\dots\le i_m=k} \frac{q^{i_1}}{1-q^{i_1}} \cdots \frac{q^{i_m}}{1-q^{i_m}} = \sum_{1\le k\le n} \frac{q^{k(m-1)}}{(1-q^k)^m}.$$

References

[1] G. Andrews, D. Crippa, and K. Simon. q–series arising from the study of random graphs. *SIAM Journal on Discrete Mathematics*, 10:41–56, 1997.

[2] G. Andrews. *The Theory of Partitions*, volume 2 of *Encyclopedia of Mathematics and its Applications*. Addison–Wesley, 1976.

[3] K. Dilcher. Some q–series identities related to divisor functions. *Discrete Mathematics*, 145:83–93, 1995.

[4] P. Flajolet and R. Sedgewick. Mellin transforms and asymptotics: Finite differences and Rice's integrals. *Theoretical Computer Science*, 144:101–124, 1995.

[5] V. Hernández. Solution IV of problem 10490 (a reciprocal summation identity). *American Mathematical Monthly*, 106:589–590, 1999.

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