A q–ANALOGUE OF A FORMULA OF HERNANDEZ OBTAINED BY INVERTING A RESULT OF DILCHER

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Abstract. We prove a q–analogue of the formula

$$\sum_{1\le k\le n}\binom{n}{k}(-1)^{k-1}\sum_{1\le i_1\le i_2\le\dots\le i_m=k}\frac{1}{i_1i_2\dots i_m}=\sum_{1\le k\le n}\frac{1}{k^m}$$

by inverting a formula due to Dilcher.

1. The identities

Hernández in [6] proved the following identity:

$$\sum_{1\le k\le n}\binom{n}{k}(-1)^{k-1}\sum_{1\le i_1\le i_2\le\dots\le i_m=k}\frac{1}{i_1i_2\dots i_m}=\sum_{1\le k\le n}\frac{1}{k^m}. \tag{1}$$

However this identity does not really require a proof, since we will show that it is just an inverted form of an identity of Dilcher [3];

$$\sum_{1\le k\le n}\binom{n}{k}(-1)^{k-1}\frac{1}{k^m}=\sum_{1\le i_1\le i_2\le\dots\le i_m\le n}\frac{1}{i_1i_2\dots i_m}. \tag{2}$$

Define for $k\ge 1$

$$a_k:=-\sum_{1\le i_1\le i_2\le\dots\le i_m=k}\frac{1}{i_1i_2\dots i_m}\qquad\text{and}\qquad b_k:=\frac{1}{k^m},$$

and $a_0=b_0=0$, then the identities are

$$\begin{aligned}\sum_{0\le k\le n}\binom{n}{k}(-1)^k a_k&=\sum_{0\le k\le n}b_k,\\ \sum_{0\le k\le n}\binom{n}{k}(-1)^k b_k&=\sum_{0\le k\le n}a_k.\end{aligned} \tag{3}$$

They are *inverse relations*, as can be seen by introducing ordinary generating functions $A(z)=\sum a_nz^n$ and $B(z)=\sum b_nz^n$. Then they are

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$$\frac{1}{1-z}A\Big(\frac{z}{z-1}\Big)=\frac{1}{1-z}B(z),$$
$$\frac{1}{1-z}B\Big(\frac{z}{z-1}\Big)=\frac{1}{1-z}A(z).$$

However

$$w=\frac{z}{z-1}\longleftrightarrow z=\frac{w}{w-1},$$

and the proof is finished. An alternative argument that will be useful in the sequel when we do the q–analogue, is as follows. We assume $a_0=b_0=0$ and take differences in (3) of the lines indexed with n resp. $n-1$; then we have to prove that

$$b_n=\sum_{1\le k\le n}\binom{n-1}{k-1}(-1)^k a_k \quad\Longleftrightarrow\quad a_n=\sum_{1\le k\le n}\binom{n-1}{k-1}(-1)^k b_k.$$

Now in this form this is a traditional inverse relation; see e. g. [7]. An explicit argument will follow in the next section for the q–instance.

We note that Dilcher's sum appears also in disguised form in [4].

2. A q–analogue

Dilcher's formula (2) was only a corollary of his elegant q–version;

$$\sum_{1\le k\le n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^{k-1}\frac{q^{\binom{k+1}{2}+(m-1)k}}{(1-q^k)^m}=\sum_{1\le i_1\le i_2\le\cdots\le i_m\le n}\frac{q^{i_1}}{1-q^{i_1}}\cdots\frac{q^{i_m}}{1-q^{i_m}}.$$

Here, $\begin{bmatrix}n\\k\end{bmatrix}_q$ denotes the Gaussian polynomial

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

with

$$(x;q)_n:=(1-x)(1-xq)\dots(1-xq^{n-1}).$$

Apart from Dilcher's paper [3], the article [2] is also of some relevance in this context.

Therefore it is a natural question to find a q–analogue of Hernández' formula, or, what amount to the same, to find the appropriate inverse relations for the q–analogues.

We state them in the following lemma.

Lemma 1. *Assume that $a_0=b_0=0$, then*

$$\begin{aligned}\sum_{1\le k\le n}b_k&=\sum_{1\le k\le n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^kq^{\binom{k}{2}}a_k,\\ \sum_{1\le k\le n}q^{-k}a_k&=\sum_{1\le k\le n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^kq^{-kn+\binom{k}{2}}b_k.\end{aligned}\tag{4}$$

Proof. First note that it is much easier to prove this than to find it (it took me a few hours, but I am not too experienced).

After a first version of this note was circulated, O. Warnaar kindly informed me that this lemma would follow from results in [5]. Nevertheless, it is perhaps instructive to provide a direct and independent proof.

Taking again differences in (4), we have to prove that

$$b_n = \sum_{1\le k\le n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} a_k \iff a_n = \sum_{1\le k\le n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^k q^{-(n-1)k+\binom{k}{2}} b_k. \tag{5}$$

Now

$$\begin{aligned}
&\sum_{1\le k\le n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^k q^{-(n-1)k+\binom{k}{2}} b_k \\
&\quad = \sum_{1\le k\le n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^k q^{-(n-1)k+\binom{k}{2}} \sum_{1\le l\le k} \begin{bmatrix} k-1 \\ l-1 \end{bmatrix}_q (-1)^l q^{\binom{l}{2}} a_l \\
&\quad = \sum_{1\le l\le n} (-1)^l q^{\binom{l}{2}} a_l \sum_{l\le k\le n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} k-1 \\ l-1 \end{bmatrix}_q (-1)^k q^{-(n-1)k+\binom{k}{2}} = a_n,
\end{aligned}$$

since for $l < n$ the inner sum is

$$\begin{aligned}
&\sum_{l\le k\le n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} k-1 \\ l-1 \end{bmatrix}_q (-1)^k q^{-(n-1)k+\binom{k}{2}} \\
&\quad = (-1)^l q^{-(n-1)l+\binom{l}{2}} \begin{bmatrix} n-1 \\ l-1 \end{bmatrix}_q \sum_{0\le m\le n-l} (-1)^m \left(q^{-(n-l-1)}\right)^m q^{\binom{m}{2}} \begin{bmatrix} n-l \\ m \end{bmatrix}_q \\
&\quad = (-1)^l q^{-(n-1)l+\binom{l}{2}} \begin{bmatrix} n-1 \\ l-1 \end{bmatrix}_q \left(q^{-(n-l-1)}; q\right)_{n-l} = 0;
\end{aligned}$$

here we have used the following version of the q–binomial theorem (see e. g. [1]):

$$\sum_{0\le m\le n} (-1)^m z^m q^{\binom{m}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q = (z; q)_n.$$

The other direction is very similar. (For our application "Dilcher" $\Rightarrow$ "Hernández" we would need only one implication.) □

We would like to remark that an alternative formulation can be given in terms of matrices of *connection coefficients*.

This can be done in terms of the original formulæ (4), but looks much nicer when referring to (5):

Define matrices

$$T := \left[\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} \right]_{n,k}, \qquad U := \left[\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^k q^{-(n-1)k+\binom{k}{2}} \right]_{n,k},$$

then

$$TU = I.$$

Theorem 2. [q–analogue of Hernández' formula]

$$\sum_{1\le k\le n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{k-1} q^{-kn+\binom{k}{2}} \sum_{1\le i_1\le i_2\le\cdots\le i_m=k} \frac{q^{i_1}}{1-q^{i_1}} \cdots \frac{q^{i_m}}{1-q^{i_m}} = \sum_{1\le k\le n} \frac{q^{k(m-1)}}{(1-q^k)^m}.$$

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