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NOTE

ON THE INTERPOLATION OF D0L-SEQUENCES

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Abstract. For a D0L-sequence $(h^n(w))_{n=0}^{\infty}$ a method is described to see this sequence as a function $\alpha \in \mathbf{R}^+ \mapsto h^\alpha(w) \in \mathcal{U}'$, where $\mathcal{U}' \supseteq \Sigma^*$ is a convenient structure.

Let Σ^* be the free monoid with unit ε generated by Σ and $\binom{x}{y}$ the binomial coefficients for words [1, 3, 4, 5, 6, 7] (i.e. the number of representations of $x = x_0y_1x_1 \cdots y_nx_n$, where $y = y_1 \cdots y_n$, $y_i \in \Sigma$).

Let

$$\mathcal{U} = \left\{ \sum_{w \in \Sigma^*} a_w w \,\middle|\, a_w \in \mathbf{R},\ a_\varepsilon = 1 \right\},$$

i.e. the subset of $\mathbf{R}\langle\langle\Sigma^*\rangle\rangle$ with $(a, \varepsilon) = 1$ (notation from [9]). In [5],[1] among other things, the following is shown: $\mathcal{U}$ with the Cauchy product is a group, the equation $\xi^n = w$, $n \in \mathbf{N}$, $w \in \mathcal{U}$ has a unique solution, and it is possible to make $\mathcal{U}$ to a topological space by means of the product topology of $\mathbf{R}$. The mappings $(x, y) \mapsto xy$, $x \mapsto x^{-1}$, $x \mapsto x^{1/n}$ are continuous. Furthermore Σ^* can be embedded into $\mathcal{U}$ by means of the mapping $w \mapsto \sum_{z \in \Sigma^*} \binom{w}{z} z$. Here it is convenient to allow the coefficients to be complex numbers; the same statements are valid. Let $\mathcal{U}'$ be obtained by replacing $\mathbf{R}$ by $\mathbf{C}$ in the definition of $\mathcal{U}$.

In [3, 4, 8] it is implicitly shown that for a given homomorphism h and given words w, z

$$\binom{h^n(w)}{z} = \eta_1^t M^n \eta_2$$

where η_1, η_2 are $(m \times 1)$-vectors and M is a $(m \times m)$-matrix with entries in $\mathbf{N}_0$. (m, η_1, η_2, M depending on h, w, z.)

Now let be $\alpha \geq 0$ and $z \in \mathbf{C}$. z^α is defined by $|z|^\alpha \, e^{i \operatorname{Arg} z \alpha}$.

[1] Available from the author; submitted for publication.

The theory of functions of matrices [2; Ch. V] yields a representation of $f(M)$, where M is a $n\times n$-matrix and $f:\mathbf{C}\to\mathbf{C}$ by

$$f(M)=\sum_{k=1}^{s}[f(\lambda_k)Z_{k1}+f'(\lambda_k)Z_{k2}+\cdots+f^{(m_k-1)}(\lambda_k)Z_{km_k}]$$

provided the derivatives exist. Here Z_{kj} are independent from f and λ_i are the eigenvalues of M with multiplicity m_i. Thus with $f_\alpha(z)=z^\alpha$ the matrix M^α can be defined and

$$\binom{h^\alpha(w)}{z}:=\eta_1^t M^\alpha \eta_2.$$

From the above representation of M^α it can be concluded that $\alpha\mapsto M^\alpha$ is continuous. Hence $\alpha\mapsto\binom{h^\alpha(w)}{z}$ is continuous for each z, and this means that $\alpha\in\mathbf{R}^+\mapsto h^\alpha(w)\in\mathcal{U}'$ is continuous.

Remark that the Cayley-Hamilton theorem gives a representation

$$M^n=\sum_{i=1}^{s}P_i(n)\lambda_i^n$$

where the P_i's are polynomials with matrices as coefficients and the λ_i's are eigenvalues. By a continuity argument

$$M^\alpha=\sum_{i=1}^{s}P_i(\alpha)\lambda_i^\alpha$$

is obtained.

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