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Representing derivatives of Chebyshev polynomials by Chebyshev polynomials and related questions

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Abstract: A recursion formula for derivatives of Chebyshev polynomials is replaced by an explicit formula. Similar formulæ are derived for scaled Fibonacci numbers.

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MSC: 11B39

1 Introduction

Consider the Chebyshev polynomials of the second kind

$$U_n(x) = \sum_{0\le k\le n/2} (-1)^k \binom{n-k}{k} (2x)^{n-2k};$$

the main interest of the paper [4] is to represent the derivatives of $U_n(x)$ in terms of the Chebyshev polynomials themselves. To this aim an "exact computational method" (a recursion formula) was presented. In the present note, we replace this "computational method" by an exact and explicit formula.

Our answer is

$$U_n^{(s)}(x) = 2^s \sum_{0\le j\le (n-s)/2} (n-j)^{\underline{s-1}} \binom{s+j-1}{s-1} (n-2j-s+1) U_{n-s-2j}(x).$$

Although it is not needed, we briefly mention without proof an analogous formula for the Chebyshev polynomials of the first kind: Let

$$T_n(x) = \sum_{0\le k\le n/2} (-1)^k \frac{n}{n-k} \binom{n-k}{k} 2^{n-1-2k} x^{n-2k},$$

then

$$T_n^{(s)}(x) = 2^s \sum_{0\le j\le (n-s)/2} n(n-1-j)^{\underline{s-1}} \binom{s+j-1}{s-1} T_{n-s-2j}(x)$$
$$- [\![n-s \text{ even}]\!]\, 2^{s-1} n\, ((n+s)/2-1)^{\underline{s-1}} \binom{(n+s)/2-1}{s-1}.$$

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We use here the notion of falling factorials $x^{\underline{n}} := x(x-1)\dots(x-n+1)$ and Iverson's symbol $[\![P]\!]$ which is 1 if P is true and 0 otherwise, compare [1].

In a last section, we turn our attention to two other families of polynomials (scaled Fibonacci numbers).

2 The proof

Our starting point is the inversion formula (see [3])

$$x^j = 2^{-j} \sum_{0\le h\le j/2} \left[\binom{j}{h} - \binom{j}{h-1} \right] U_{j-2h}(x),$$

which we will use in

$$U_n^{(s)}(x) = \sum_{0\le k\le n/2} (-1)^k \binom{n-k}{k} (n-2k)^{\underline{s}} 2^{n-2k} x^{n-s-2k}$$

and simplify:

$$\begin{aligned}
U_n^{(s)}(x) &= \sum_{0\le k\le n/2} (-1)^k \binom{n-k}{k} (n-2k)^{\underline{s}} 2^s \\
&\qquad \sum_{k\le h+k\le (n-s)/2} \left[\binom{n-s-2k}{h} - \binom{n-s-2k}{h-1} \right] U_{n-s-2k-2h}(x) \\
&= \sum_{0\le k\le n/2} (-1)^k \binom{n-k}{k} (n-2k)^{\underline{s}} 2^s \sum_{k\le j\le (n-s)/2} \left[\binom{n-s-2k}{j-k} - \binom{n-s-2k}{j-k-1} \right] U_{n-s-2j}(x) \\
&= 2^s \sum_{0\le k\le j\le (n-s)/2} (-1)^k \binom{n-k}{k} (n-2k)^{\underline{s}} \left[\binom{n-s-2k}{j-k} - \binom{n-s-2k}{j-k-1} \right] U_{n-s-2j}(x). \qquad (1)
\end{aligned}$$

We compute the sum over k separately:

$$\begin{aligned}
&\sum_{0\le k\le j} (-1)^k \binom{n-k}{k} (n-2k)^{\underline{s}} \left[\binom{n-s-2k}{j-k} - \binom{n-s-2k}{j-k-1} \right] \\
&\quad = \sum_{0\le k\le j} (-1)^k \left[\frac{(n-k)!}{k!(j-k)!(n-s-k-j)!} - \frac{(n-k)!}{k!(j-1-k)!(n-s-k-j+1)!} \right] \\
&\quad = \sum_{0\le k\le j} (-1)^k \left[\frac{(n-j)!}{(n-s-j)!} \binom{n-k}{j-k} \binom{n-s-j}{k} - \frac{(n-j+1)!}{(n-s-j+1)!} \binom{n-k}{j-1-k} \binom{n-s-j+1}{k} \right] \\
&\quad = \sum_{0\le k\le j} (-1)^j \left[(n-j)^{\underline{s}} \binom{-n+j-1}{j-k} \binom{n-s-j}{k} + (n-j+1)^{\underline{s}} \binom{-n+j-2}{j-1-k} \binom{n-s-j+1}{k} \right] \\
&\quad = (-1)^j \left[(n-j)^{\underline{s}} \binom{-1-s}{j} + (n-j+1)^{\underline{s}} \binom{-1-s}{j-1} \right] \\
&\quad = (n-j)^{\underline{s}} \binom{s+j}{j} - (n-j+1)^{\underline{s}} \binom{s+j-1}{j-1} \\
&\quad = (n-j)^{\underline{s-1}} \binom{s+j-1}{s-1} \left[(n-j-s+1)\frac{s+j}{s} - (n-j+1)\frac{j}{s} \right] \\
&\quad = (n-j)^{\underline{s-1}} \binom{s+j-1}{s-1} (n-2j-s+1).
\end{aligned}$$

In this computation only the Vandermonde convolution formula [1] was used.

Plugging this formula into (1) yields the announced formula from the introduction.

3 Scaled Fibonacci numbers: a similar analysis

In the very recent paper [2], the following polynomials have been investigated:

$$\hat{a}_n(\delta) = \sum_{k=0}^{n} \binom{n}{k} F_{k-1} (-\delta)^k,$$
$$\hat{b}_n(\delta) = -\sum_{k=1}^{n} \binom{n}{k} F_k (-\delta)^k.$$

For our purposes, the n-th polynomial should have degree n. Therefore we consider the following slight variations:

$$a_n(x) = \sum_{k=0}^{n} \binom{n}{k} F_{k+1} (-x)^k,$$
$$b_n(x) = \sum_{k=1}^{n+1} \binom{n+1}{k} F_k (-x)^{k-1}.$$

Our goal, as before, is to express the derivatives of the polynomials by the polynomials themselves. Since both families are a basis for the vector space of polynomials, this can be achieved in a unique way. We will work out the corresponding coefficients in the sequel. Although it is not needed, we give the double generating functions:

$$\sum_{n\ge 0} t^n a_n(x) = \frac{1-t}{1+(x-2)t+(1-x-x^2)t^2},$$
$$\sum_{n\ge 0} t^n b_n(x) = \frac{1}{1+(x-2)t+(1-x-x^2)t^2}.$$

To check this is simple:

$$\sum_{n\ge 0} t^n a_n(x) = \sum_{n\ge k\ge 0} t^n \binom{n}{k} F_{k+1} (-x)^k = \sum_{k\ge 0} \frac{t^k}{(1-t)^{k+1}} F_{k+1} (-x)^k$$
$$= \frac{1}{1-t} \frac{1}{1-z-z^2}\Bigg|_{z=\frac{-tx}{1-t}} = \frac{1-t}{1+(x-2)t+(1-x-x^2)t^2},$$

as predicated, the other formula being similar.

Now we need to invert: we seek the unique coefficients such that

$$\sum_{k=0}^{n} c_{n,k} a_k(x) = x^n, \qquad \sum_{k=0}^{n} d_{n,k} b_k(x) = x^n.$$

They are given by $c_{n,k} = \frac{1}{F_{n+1}} (-1)^k \binom{n}{k}$ and $d_{n,k} = \frac{1}{F_{n+1}} (-1)^k \binom{n+1}{k+1}$.

The proofs that this works are straightforward:

$$\sum_{k=0}^{n} \frac{1}{F_{n+1}} (-1)^k \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} F_{l+1} (-x)^l$$
$$= \frac{1}{F_{n+1}} \sum_{l=0}^{n} F_{l+1} (-x)^l \binom{n}{l} \sum_{k=l}^{n} (-1)^k \binom{n-l}{k-l}$$
$$= \frac{1}{F_{n+1}} \sum_{l=0}^{n} F_{l+1} (-x)^l \binom{n}{l} (-1)^l [\![n=l]\!] = x^n.$$

The other one is similar:

$$\sum_{k=0}^{n} \frac{1}{F_{n+1}} (-1)^k \binom{n+1}{k+1} \sum_{l=1}^{k+1} \binom{k+1}{l} F_l (-x)^{l-1}$$

$$= \frac{1}{F_{n+1}} \sum_{l=1}^{n+1} F_l (-x)^{l-1} \binom{n+1}{l} \sum_{k=l-1}^{n} (-1)^k \binom{n+1-l}{k+1-l}$$
$$= \frac{1}{F_{n+1}} \sum_{l=1}^{n+1} F_l (-x)^{l-1} \binom{n+1}{l} (-1)^{l-1} [\![l = n+1]\!] = x^n.$$

Now we consider the s-th derivative:

$$\begin{aligned}
a_n^{(s)}(x) &= \sum_{k=s}^{n} \binom{n}{k} F_{k+1} (-1)^k \frac{k!}{(k-s)!} x^{k-s} \\
&= \sum_{k=s}^{n} \binom{n}{k} F_{k+1} (-1)^k \frac{k!}{(k-s)!} \sum_{l=0}^{k-s} c_{k-s,l} a_l(x) \\
&= \sum_{k=0}^{n-s} \binom{n}{k+s} F_{k+s+1} (-1)^{k+s} \frac{(k+s)!}{k!} \sum_{l=0}^{k} c_{k,l} a_l(x) \\
&= \sum_{l=0}^{n-s} a_l(x) \sum_{k=l}^{n-s} \binom{n}{k+s} F_{k+s+1} (-1)^{k+s} \frac{(k+s)!}{k!} \frac{1}{F_{k+1}} (-1)^l \binom{k}{l} \\
&= s! \sum_{l=0}^{n-s} a_l(x) (-1)^l \binom{n}{l} \binom{n-l}{s} \sum_{k=l}^{n-s} \binom{n-s-l}{k-l} \frac{F_{k+s+1}}{F_{k+1}} (-1)^{k+s} \\
&= s! (-1)^s \sum_{l=0}^{n-s} a_l(x) \binom{n}{l} \binom{n-l}{s} \sum_{k=0}^{n-s-l} \binom{n-s-l}{k} \frac{F_{k+l+s+1}}{F_{k+l+1}} (-1)^k.
\end{aligned}$$

In order to simplify the inner sum, let us write $N := n-s-l$ and assume that $N \geq 1$, since the instance $N = 0$ is different (and trivial). Notice that

$$\sum_{k=0}^{N} \binom{N}{k} (-1)^k = 0.$$

Then

$$\begin{aligned}
\sum_{k=0}^{N} \binom{N}{k} (-1)^k \frac{F_{k+l+s+1}}{F_{k+l+1}} &= \sum_{k=0}^{N} \binom{N}{k} (-1)^k \frac{F_{k+l+1} F_{s+1} + F_{k+l} F_s}{F_{k+l+1}} \\
&= \sum_{k=0}^{N} \binom{N}{k} (-1)^k \left[F_{s+1} + F_s \frac{F_{k+l}}{F_{k+l+1}} \right] \\
&= F_s \sum_{k=0}^{N} \binom{N}{k} (-1)^k \frac{F_{k+l}}{F_{k+l+1}}.
\end{aligned}$$

We leave it as a challenge to simplify this N-th difference even further.

And now we differentiate the other polynomials:

$$\begin{aligned}
b_n^{(s)}(x) &= \sum_{k=s+1}^{n+1} \binom{n+1}{k} F_k (-1)^{k-1} \frac{(k-1)!}{(k-1-s)!} x^{k-1-s} \\
&= \sum_{k=s+1}^{n+1} \binom{n+1}{k} F_k (-1)^{k-1} \frac{(k-1)!}{(k-1-s)!} \sum_{l=0}^{k-1-s} d_{k-1-s,l} b_l(x) \\
&= \sum_{k=0}^{n-s} \binom{n+1}{k+s+1} F_{k+s+1} (-1)^{k+s} \frac{(k+s)!}{k!} \sum_{l=0}^{k} d_{k,l} b_l(x) \\
&= \sum_{l=0}^{n-s} b_l(x) \sum_{k=l}^{n-s} \binom{n+1}{k+s+1} F_{k+s+1} (-1)^{k+s} \frac{(k+s)!}{k!} \frac{1}{F_{k+1}} (-1)^l \binom{k+1}{l+1}
\end{aligned}$$

$$= s! \sum_{l=0}^{n-s} b_l(x)(-1)^l \binom{n+1}{l+1} \binom{n-l}{s} \sum_{k=l}^{n-s} \frac{k+1}{k+s+1} \binom{n-s-l}{k-l} \frac{F_{k+s+1}}{F_{k+1}} (-1)^{k+s}$$

$$= s!(-1)^s \sum_{l=0}^{n-s} b_l(x)(-1)^l \binom{n+1}{l+1} \binom{n-l}{s} \sum_{k=0}^{n-s-l} \frac{k+l+1}{k+l+s+1} \binom{n-s-l}{k} \frac{F_{k+l+s+1}}{F_{k+l+1}} (-1)^k.$$

The inner sum can be reduced to the computation of

$$\sum_{k=0}^{N} \binom{N}{k} (-1)^k \frac{k+l+1}{k+l+s+1} \frac{F_{k+l}}{F_{k+l+1}},$$

but we have little hope that this can be turned into something nice.

References

[1] Graham R.L., Knuth D.E., Patashnik O., Concrete Mathematics, 2nd ed., Addison-Wesley, Reading, 1994.

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