The Asymmetric Leader Election Algorithm with swedish stopping: A probabilistic analysis

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Dedicated to the memory of Philippe Flajolet.

Abstract

We study a leader election protocol that we call the Swedish leader election protocol. This name comes from a protocol presented by L. Bondesson, T. Nilsson, and G. Wikstrand in [1]. The goal is to select one among $n > 0$ players, by proceeding through a number of rounds. If there is only one player remaining, the protocol stops and the player is declared the leader. Otherwise, all remaining players flip a biased coin; with probability q the player survives to the next round, with probability $p = 1 - q$ the player loses (is killed) and plays no further ... unless all players lose in a given round (null round), so all of them play again. In the classical leader election protocol, any number of *null* rounds may take place, and with probability 1 some player will ultimately be elected. In the Swedish leader election protocol there is a maximum number τ of consecutive null rounds, and if the threshold is attained the protocol fails without declaring a leader.

In this paper, several parameters are asymptotically analyzed, among them: Success Probability, Number of rounds K_n, Number of null rounds T_n.

This paper is a companion paper to [7] where De-Poissonization was used, together with the Mellin transform. While this works fine as far as it goes, there are limitations, in particular of a computational nature. The approach chosen here is similar to earlier efforts of the same authors [8, 9, 10]. Identifying some underlying distributions as Gumbel (type) distributions, one can start with approximations at a very early stage and compute (at least in principle) all moments asymptotically. This is in contrast to [7] where only expected values were considered. In an appendix, it is shown that, whereever results are given in both papers, they coincide, although they are presented in different ways.

1 Introduction

We present a probabilistic analysis, based on an urn model, of a leader election protocol that we call the Swedish leader election protocol. This name comes from a protocol presented by L. Bondesson, T. Nilsson, and G. Wikstrand in [1]. The goal is to select one among $n > 0$ players, by proceeding through a number of rounds. If there is only one player remaining, the protocol stops and the player is declared the leader. Otherwise, all remaining players flip a biased coin; with probability q the player survives to the next round, with probability $p = 1 - q$ the player loses (is killed) and plays no further ... unless all players lose in a given round (null round), so all of them play again. In the classical leader election protocol, any number of *null* rounds may take place, and with probability 1 some player will ultimately be elected. In the Swedish leader election protocol there is a maximum number τ of consecutive null rounds, and if the threshold is attained the protocol fails without declaring a leader.

In this paper, several parameters are asymptotically analyzed, starting with n players (n large):

1. Success Probability.

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2. Number of rounds K_n.

3. Number of null rounds T_n. We say that a round is *null* if every active player tosses tails (all killed).

4. Number W_n, in case of failure, of players that were active at the last non-null round, the so-called *left-overs*.

5. Total number C_n of coins flipped.

2 Urn model

This paper is a companion paper to [7] where De-Poissonization was used, together with the Mellin transform. Here, we approximate the relevant quantities at an early stage; the loss in accuracy results in a gain when it comes to the complexity of the necessary computations. Indeed, we can deal here with all moments, whereas in [7] only expected values could be computed.

Here, we will proceed as in [10], with urns numbered $1, 2, \ldots$. This urn model, together with the underlying Gumbel (type) distribution(s) is the key to the success.

Let us consider the model as a sequence of n geometric iid random variables (RVs), with distribution pq^{i-1}. Each RV corresponds to a ball thrown into urn i. The relation with the leader election algorithm can be described as follows. Each player does have some life duration (given by the geometric RV) and the players killed at step j correspond to the number of balls falling into urn j.

We state, for later use, the following properties, for large n:

- We have asymptotic independence of urns, for all events related to urn j containing $\mathcal{O}(1)$ balls. This is proved, by Poissonization-De-Poissonization, in [9], [11] and [4] (in this paper for $p = 1/2$, but the proof is easily adapted). The error term is $\mathcal{O}(n^{-C})$ where C is a positive constant.

- We obtain asymptotic distributions of the interesting RVs. The number of balls in each urn is now Poisson-distributed with parameter npq^{j-1} in urn j containing $\mathcal{O}(1)$ balls. The asymptotic distibutions are related to Gumbel distribution functions or convergent series of such. The error term is $\mathcal{O}(n^{-1})$.

- Some summations now go to ∞. This is justified, for example, in [9].

- We have uniform integrability for the moments of our RVs. To show that the limiting moments are equivalent to the moments of the limiting distributions, we need a suitable rate of convergence. This is related to a uniform integrability condition (see Loève [6, Section 11.4]). For the kind of limiting distributions we consider here, the rate of convergence is analyzed in detail in [8] and [11]. The error term is $\mathcal{O}(n^{-C})$.

- Asymptotic expressions for the moments are obtained by Mellin transforms. The error term is $\mathcal{O}(n^{-C})$.

- $\Gamma(s)$ decreases exponentially in the direction $i\infty$:

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}.$$

  Also, we have a "slow increase property" for all other functions we encounter. So inverting the Mellin transforms is easily justified.

We proceed as follows: from the asymptotic properties of the urns, we obtain the asymptotic distributions of our RV of interest. Next we compute the Laplace transform $\phi(\alpha)$ from which we can derive the dominant part of probabilities and moments as well as the (tiny) periodic part in the form of a Fourier series. Note that we will also need the first values of probabilities and moments, obtained via some recurrences.

If we compare the approach in this paper with other ones that appeared previously, then we can notice the following. Traditionally, one would stay with exact enumerations as long as possible, and only at a late stage move to asymptotics. Doing this, one would, in terms of asymptotics, carry many unimportant contributions around, which makes the computations quite heavy, especially when it comes to higher moments. Here, however, approximations are carried out as early as possible, and this allows for streamlined (and often automatic) computations of asymptotic distributions and higher moments.

As suggested by one referee, other RVs of interest could be considered, for instance the number of useless coin flips (i.e. coin flips that lead to null rounds or rounds where every player goes forward to the next round). These RVs can indeed be analyzed with the same techniques, but due to length constraints, we refrain to do this here. Such considerations would make an excellent project for research students.

The present paper falls into the paradigm of *combinatorics of geometrically distributed words*; many papers have been written on that since the mid-nineties. We cite only the first one in this series: [12]

The paper is organized as follows: Section 3 presents our main notations; more detailed notations will be presented at the beginning of each section. Section 4 is devoted to the success probability, Section 5 to 8 give the asymptotic distribution and first two moments of the RVs of interest (all moments can be derived by the same technique): we compute the dominant and periodic part, both in the success and failure case. Appendices A summarizes some definitions and identities. In [7], was presented an analytic treatment of the Swedish leader election protocol. In particular, the success probability and the dominant part of the mean of the following RV was computed: total number of rounds, total number of null rounds, number of left-overs. We prove, in Appendices B to E, the equivalence with the results given in this paper. Some matrix expressions are presented in Appendix F and Appendix G briefly presents the second model where we fail if we have τ null records, consecutive or not. Appendix H gives the proof of Theorem 8.2.

3 Notations

We will use several abbreviations for probabilities and moments in order to derive more compact expressions.

$$\begin{aligned}
n &:= \text{number of initial players}, n \text{ large},\\
\mathcal{P}(\lambda,u) &:= e^{-\lambda}\lambda^u/u!,\ \text{(Poisson distribution)},\\
n^* &:= n\frac{p}{q},\\
Q &:= 1/q,\\
\log &:= \log_Q,\\
\eta &:= j-\log n^* \quad \text{or} \quad \eta := k-\log n^*,\\
L &:= \ln Q,\\
\{x\} &:= \text{ fractional part of } x,\\
\tilde{\alpha} &:= \alpha/L,\\
M &:= \log p,\\
\chi_l &:= \frac{2l\pi\mathbf{i}}{L},\\
K_i &:= \text{total number of rounds, starting with } i \text{ players},\\
T_i &:= \text{total number of null rounds, starting with } i \text{ players},\\
W_i &:= \text{total number of leftovers, starting with } i \text{ players},\\
C_i &:= \text{total number of flipped coins, starting with } i \text{ players},
\end{aligned}$$

$$
\begin{aligned}
R_n &:= \mathbb{E}(K_n),\\
I_n &:= \mathbb{E}(T_n),\\
L_n &:= \mathbb{E}(W_n),\\
N_n &:= \mathbb{E}(C_n),\\
I &:= \text{number of balls in the maximal non-empty urn,}\\
J &:= \text{either the position of the maximal non-empty urn, if it contains } I > 1 \text{ balls,}\\
&\quad \text{or the position of the last non-empty urn \emph{before} the maximal non-empty urn,}\\
&\quad \text{if the latter contains } I = 1 \text{ ball,}\\
\Sigma_1(i,\tau) &:= \sum_{v=0}^{\tau-1}(p^i)^v = \frac{1-p^{i\tau}}{1-p^i},\\
\Sigma_2(i,\tau) &:= \sum_{v=0}^{\tau-1}(p^i)^v(v+1) = \frac{p^{i\tau}(-1+\tau p^i-\tau)+1}{(1-p^i)^2},\\
\Sigma_3(i,\tau) &:= \sum_{v=0}^{\tau-1}(p^i)^v(v+1)^2 = \frac{p^{i\tau}(p^i+1-2\tau p^i+2\tau+\tau^2p^{2i}-2\tau^2p^i+\tau^2)-p^i+1}{(1-p^i)^3},\\
\Sigma_4(i,\tau) &:= \sum_{v=0}^{\tau-1}(p^i)^v v = \frac{p^{i\tau}(\tau p^i-\tau-p^i)+p^i}{(1-p^i)^2},\\
\Sigma_5(i,\tau) &:= \sum_{v=0}^{\tau-1}(p^i)^v v^2 = \frac{p^{i\tau}(p^i+2\tau p^i+\tau^2p^{2i}-2\tau^2p^i+\tau^2+p^{2i}-2\tau p^{2i})-p^{2i}+p^i}{(1-p^i)^3}.
\end{aligned}
$$

$$
\begin{aligned}
&\text{Model 1} = \text{we fail if we have } \tau \text{ consecutive null records,}\\
&\text{Model 2} = \text{we fail if we have } \tau \text{ null records, consecutive or not.}
\end{aligned}
$$

For any of the four RVs, we will denote by $\xi_i^{(2)}$, $\xi \in \{R, I, L, F\}$ the expectation of the square of the corresponding RVs.

Note that the maximal non-empty urn, if it contains $i \geq 2$ balls, corresponds to a null round. If the maximal non-empty urn contains $i = 1$ ball, the process is successful.

We will first analyze the Model 1: we fail if we have τ consecutive null records. The Model 2: we fail if we have τ null records, consecutive or not, will be briefly considered in Appendix G.

4 Success probability

The notations we use in this section can be summarized as follows:[1]

$$
\begin{aligned}
S_i = P(i) &:= \text{Probability that, starting with } i \text{ players, we succeed,}\\
\tilde{P}(i) &:= \text{Probability that, starting with } i \text{ players, we succeed,}\\
&\quad \text{given that the } i \text{ players were obtained in a null round, not preceded by another null round,}\\
S_n &:= \text{success probability, starting with } n \text{ players,}\\
F_n &:= \text{failure probability, starting with } n \text{ players} = 1 - S_n.
\end{aligned}
$$

Note that we need to distinguish between starting with i players (used in the recurrences and the asymptotics) and n which denotes the initial number of players.

[1]The tilde notation introduced here will always have the same meaning in the sequel.

We will first show how to compute the basic probabilities, which will be used in the sequel. Next, as explained in Section 2, we proceed to some asymptotic distributions, compute their Laplace transform and finally derive the asymptotic expressions for distributions.

We have the following recurrences:

$$\begin{aligned}
P(1) &= 1, \quad \tilde{P}(1) = 1, \\
P(i) &= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P(\ell) = \Sigma_1(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P(\ell), \ i \geq 2, \\
\tilde{P}(i) &= \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P(\ell) = \Sigma_1(i,\tau-1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P(\ell) \\
&= \frac{\Sigma_1(i,\tau-1)}{\Sigma_1(i,\tau)} P(i), \ i \geq 2.
\end{aligned}$$

Explanation: We can have up to $\tau - 1$ or $\tau - 2$ null rounds (all killed), followed by ℓ survivors. Note that $\ell = i$ in the right-hand side leads to $P(i)$.

We recall that we have asymptotic independence of urns, for all events related to urn j containing $\mathcal{O}(1)$ balls. Also the number of balls in each such urn is now Poisson-distributed with parameter npq^{j-1} in urn j.

We also have from [10], (here and in the sequel $\sim$ always denotes $\sim_{n\to\infty}$), with $\eta := j - \log n^*$,

$$\mathbb{P}(J = j, I = i) \sim \exp\Big(-\frac{1}{p} e^{-L\eta}\Big) \frac{e^{-L\eta i}}{i!}, \ i \geq 2, \tag{1}$$

$$\begin{aligned}
\mathbb{P}(J = j, I \geq 2, S) &\sim f_1(\eta), \\
f_1(\eta) &:= \sum_{i=2}^{\infty} \exp\Big(-\frac{1}{p} e^{-L\eta}\Big) \frac{e^{-L\eta i}}{i!} \tilde{P}(i), \\
\mathbb{P}(J = j, I = 1) &\sim f_2(\eta), \\
f_2(\eta) &:= \exp\Big(-\frac{q}{p} e^{-L\eta}\Big) \frac{q}{p} e^{-L\eta} \left(1 - \exp\left(-e^{-L\eta}\right)\right).
\end{aligned} \tag{2}$$

Explanation:

The asymptotic number ℓ of balls in urn j is given by

$$\exp\left(-npq^{j-1}\right) \frac{\left(npq^{j-1}\right)^\ell}{\ell!},$$

and with $\eta = j - \log n^*$, this is equivalent to $\mathcal{P}\left(e^{-L\eta}, \ell\right)$. This leads to

$$\begin{aligned}
\mathbb{P}(J = j, I = i) &\sim \mathcal{P}\Big(\frac{q}{p} e^{-L\eta}, 0\Big) \mathcal{P}\left(e^{-L\eta}, i\right), \ i \geq 2, \\
\mathbb{P}(J = j, I = 1) &\sim \mathcal{P}\Big(\frac{q}{p} e^{-L\eta}, 1\Big) \left[1 - \mathcal{P}\left(e^{-L\eta}, 0\right)\right].
\end{aligned}$$

Note that the case $i \geq 2$ does not necessarily lead to a success: urn J corresponds to the first null round, hence the multiplication by $\tilde{P}(i)$. On the other hand, the case $i = 1$ does lead to a success: urn J corresponds to a round with one single player alive, which is immediately declared as the leader (there are no null rounds before).

We now compute the Laplace transform. This gives

$$\phi_1(\alpha) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_1(\eta) d\eta = \sum_{i=2}^{\infty} \frac{(1/p)^{-i+\tilde{\alpha}}}{Li!} \Gamma(i - \tilde{\alpha}) \tilde{P}(i).$$

Again from [10] the corresponding dominant part of S_n is given by

$$\phi_1(0) = \sum_{i=2}^{\infty} \frac{p^i}{Li} \frac{\Sigma_1(i,\tau-1)}{\Sigma_1(i,\tau)} P(i).$$

The corresponding periodic part is given by

$$\omega_{1,1} = \sum_{l\neq 0} \varphi_{1,1}(\chi_l) e^{-2l\pi \mathbf{i}\{\log n^*\}},$$

with

$$\varphi_{1,1}(\chi_l) = \phi_1(\alpha)\big|_{\alpha=-L\chi_l}.$$

We obtain

$$\varphi_{1,1}(\chi_l) = \sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!} \frac{\Sigma_1(i,\tau-1)}{\Sigma_1(i,\tau)} P(i).$$

Also,

$$\phi_2(\alpha) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_2(\eta) d\eta = \frac{1}{L}\left[\left(\frac{q}{p}\right)^{\tilde{\alpha}} - q\left(\frac{1}{p}\right)^{\tilde{\alpha}}\right]\Gamma(1-\tilde{\alpha}).$$

Hence

$$\begin{aligned} \phi_2(0) &= \frac{p}{L} = \text{Probability that the maximal non-empty urn contains one ball,} \qquad (3)\\ \omega_{2,1} &= \sum_{l\neq 0} \varphi_{2,1}(\chi_l) e^{-2l\pi \mathbf{i}\{\log n^*\}},\\ \varphi_{2,1}(\chi_l) &= \frac{1}{L}\left[\left(\frac{q}{p}\right)^{-\chi_l} - q\left(\frac{1}{p}\right)^{-\chi_l}\right]\Gamma(1+\chi_l) = \frac{p^{1+\chi_l}}{L}\Gamma(1+\chi_l). \end{aligned}$$

And finally, with notations provided in the Appendix A, we have the following result.

Theorem 4.1 *Related to success probability, we have*

$$\begin{aligned} S_n &\sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \frac{\Sigma_1(i,\tau-1)}{\Sigma_1(i,\tau)} P(i) + \frac{p}{L} + \sum_{l\neq 0} \varphi_{1,1}(\chi_l) e^{-2l\pi \mathbf{i}\{\log n^*\}} + \sum_{l\neq 0} \varphi_{2,1}(\chi_l) e^{-2l\pi \mathbf{i}\{\log n^*\}}\\ &=: V_1 + \frac{p}{L} + \sum_{l\neq 0} \varphi_{1,1}(\chi_l) e^{-2l\pi \mathbf{i}\{\log n^*\}} + \sum_{l\neq 0} \varphi_{2,1}(\chi_l) e^{-2l\pi \mathbf{i}\{\log n^*\}}. \end{aligned}$$

It might look paradoxical at first sight, that the asymptotic formula involves the quantities $P(i)$ on the right site. However, that happens often, and convergence is quite good, so that with only a few terms (obtained directly from the recursion) a good approximation of the numerical values can be obtained. This phenomenon will appear in all our subsequent analyses.

Of course, $F_n = 1 - S_n$. Note also that as $\tau \to \infty$, the dominant part gives

$$\sum_{i=2}^{\infty} \frac{p^i}{Li} + \frac{p}{L} = 1$$

as expected. For further use, we denote

$$\begin{aligned} \Pi_1 &:= \frac{p}{L} \quad \text{(one ball in the maximal non-empty urn)},\\ \Pi_2(i) &:= \frac{p^i}{Li},\\ Pd(S) &= V_1 + \frac{p}{L} \quad \text{(dominant part of } S_n\text{)}. \end{aligned}$$

5 Asymptotic distribution and moments of $K_n - \log n^*$

5.1 Asymptotic distribution and moments of $K_n - \log n^*$, success case

Notations:

$$\begin{aligned}
P_K(i,k) &:= \text{Probability that, starting with } i \text{ players, we succeed after } k \text{ rounds,}\\
\tilde{P}_K(i,k) &:= \text{Probability that, starting with } i \text{ players, we succeed after } k \text{ rounds,}\\
&\quad\ \text{given that the } i \text{ players were obtained in a null round,}\\
R_i &:= \text{average number of rounds, starting with } i \text{ players, with success at the end,}\\
\tilde{R}_i &:= \text{average number of rounds, starting with } i \text{ players, with success at the end.}
\end{aligned}$$

We will first compute some asymptotic distributions, then the recurrences for the moments and finally the asymptotics for distributions and moments.

In case of success, the moments of $K_n - \log n^*$ are computed as in [10], and expressed with some $\tilde{R}_i$, $\tilde{R}_i^{(2)}$, instead of $x_{i,S}$, $x_{i,S}^{(2)}$ used in [10]. They are computed as described in the sequel. Here and in the sequel, we give the first two moments. All moments could be computed, only with more (algebraic and Maple) efforts.

We use $f_1(\eta)$ as given by (4) of [10], with $\tilde{P}_K(i,k)$ instead of $P(i,k)$ and $\eta := k - \log n^*$, i.e.,

$$f_1(\eta) = \sum_{k=1}^{\infty}\sum_{i=2}^{\infty} \exp\Big(-\frac{1}{p}e^{Lk}e^{-L\eta}\Big)\frac{e^{-L\eta i}e^{Lki}}{i!}\tilde{P}_K(i,k).$$

$f_2(\eta)$ is given by (2). Indeed, in the case $I = 1$, K is identical to J.

We have the following recurrences:

$$\begin{aligned}
P_K(1,0) &= 1, \quad \tilde{P}_K(1,0) = 1,\\
P_K(1,\geq 1) &= 0, \quad \tilde{P}_K(1,\geq 1) = 0,\\
P_K(i,k) &= \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_K(\ell,k-1-s),\ i \geq 2,\\
\tilde{P}_K(i,k) &= \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_K(\ell,k-1-s),\ i \geq 2.
\end{aligned}$$

Explanation: We can have up to $\tau - 1$ or $\tau - 2$ null rounds (all killed), followed by ℓ survivors. This takes $s + 1$ rounds already. This leads to the following recurrences. As already mentioned, these preliminary expressions will be needed in the asymptotic formulæ.

$$\begin{aligned}
R_i &= \sum_k P_K(i,k)k = \sum_k\sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_K(\ell,k-1-s)[k-1-s+s+1],\\
&= \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell + \sum_k\sum_{s=0}^{\tau-1}(p^i)^s(s+1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_K(\ell,k-1-s)\\
&= \Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell + \Sigma_2(i,\tau)P(i)/\Sigma_1(i,\tau),\ R_1 = 0,
\end{aligned}$$

$$\begin{aligned}
\tilde{R}_i &= \sum_k \tilde{P}_K(i,k)k = \sum_k\sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_K(\ell,k-1-s)[k-1-s+s+1],\\
&= \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell + \sum_k\sum_{s=0}^{\tau-2}(p^i)^s(s+1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_K(\ell,k-1-s)
\end{aligned}$$

$$= \Sigma_1(i,\tau-1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell + \Sigma_2(i,\tau-1)P(i)/\Sigma_1(i,\tau),\ \tilde{R}_1 = 0,$$

$$R_i^{(2)} = \sum_k P_K(i,k)k^2 = \sum_k\sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_K(\ell,k-1-s)[k-1-s+s+1]^2,$$

$$= \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell^{(2)} + \sum_k\sum_{s=0}^{\tau-1}(p^i)^s(s+1)^2\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_K(\ell,k-1-s)$$

$$+ 2\sum_{s=0}^{\tau-1}(p^i)^s(s+1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell$$

$$= \Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell^{(2)} + \Sigma_3(i,\tau)P(i)/\Sigma_1(i,\tau) + 2\Sigma_2(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell,\ R_1^{(2)} = 0,$$

$$\tilde{R}_i^{(2)} = \sum_k \tilde{P}_K(i,k)k^2 = \sum_k\sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_K(\ell,k-1-s)[k-1-s+s+1]^2,$$

$$= \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell^{(2)} + \sum_k\sum_{s=0}^{\tau-2}(p^i)^s(s+1)^2\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_K(\ell,k-1-s)$$

$$+ 2\sum_{s=0}^{\tau-2}(p^i)^s(s+1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell$$

$$= \Sigma_1(i,\tau-1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell^{(2)} + \Sigma_3(i,\tau-1)P(i)/\Sigma_1(i,\tau)$$

$$+ 2\Sigma_2(i,\tau-1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell,\ \tilde{R}_1^{(2)} = 0.$$

Note that, in previous expressions, we could replace

$$\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell \qquad \text{by} \qquad [R_i - \Sigma_2(i,\tau)P(i)/\Sigma_1(i,\tau)]/\Sigma_1(i,\tau).$$

To obtain the moments of $K_n - \log n^*$, we plug, *mutatis mutandis*, $\tilde{R}_i$, $\tilde{R}_i^{(2)}$ into the moments given in [10]. Note that to each value $I = i \geq 2$ corresponds $\tilde{P}(i)$ as explained in Section 4. Also $A_0(\chi_l)$ is no more null here and $\tilde{P}(1) = 1$ by convention. This leads, with the quantities defined in the Appendix A, to

Theorem 5.1 *Asymptotic distribution and moments of $K_n - \log n^*$, success case.*

$$\mathbb{P}(K_n = k) \sim f_1(\eta) + f_2(\eta),$$

$$R_n - \log n^* = \mathbb{E}(K_n - \log n^*) \sim U_1 - MV_1 - \frac{V_2}{L} + \frac{p\gamma}{L^2} - \frac{1+pM}{L}$$
$$+ \sum_{l\neq 0}\left[B_1(\chi_l) - MA_0(\chi_l) - \frac{A_1(\chi_l)}{L} - \frac{\Gamma(1+\chi_l)}{L}\right]e^{-2l\pi\mathbf{i}\{\log n\}},$$

$$\mathbb{E}(K_n - \log n^*)^2 \sim U_2 - 2MU_1 - 2\frac{U_4}{L} + M^2V_1 + 2M\frac{V_2}{L} + \frac{V_3+V_4}{L^2}$$
$$+ \frac{p(\pi^2/6+\gamma^2)}{L^3} - \frac{2\gamma(pM+1)}{L^2} + \frac{pM^2+2M+1}{L}$$
$$+ \sum_{l\neq 0}\left\{B_2(\chi_l) - 2MB_1(\chi_l) - 2\frac{B_3(\chi_l)}{L} + M^2A_0(\chi_l) + 2M\frac{A_1(\chi_l)}{L} + \frac{A_2(\chi_l)+A_3(\chi_l)}{L^2}\right.$$

$$+\Gamma(1+\chi_l)\left[2\frac{\psi(1+\chi_l)}{L^2}+\frac{1}{L}+2\frac{M}{L}\right]\Bigg\}e^{-2l\pi\mathbf{i}\{\log n\}}.$$

Note that the periodic component contains $\{\log n\}$ *in the exponent (and not* $\{\log n^*\}$*). Note also that, again, the asymptotics depend on* R_i *on the right-side.*

To obtain the moments of $K_n - \log n^*$, *given success*, we simply divide the moments given in the theorem by S_n.

5.2 Asymptotic distribution and moments of $K_n - \log n^*$, failure case

The analysis of this RV (as well as the next ones, with the exception of the number of flipped coins) follows the same pattern as the previous ones. We will only present the necessary expressions.

Notations:

$$\begin{aligned}
P'_K(i,k) &:= \text{Probability that, starting with } i \text{ players, we fail after } k \text{ rounds},\\
\tilde{P}'_K(i,k) &:= \text{Probability that, starting with } i \text{ players, we fail after } k \text{ rounds},\\
&\quad \text{given that the } i \text{ players were obtained in a null round},\\
P'(i) &:= \text{Probability that, starting with } i \text{ players, we fail} = 1-P(i),\\
\tilde{P}'(i) &:= \text{Probability that, starting with } i \text{ players, we fail} = 1-\tilde{P}(i),\\
&\quad \text{given that the } i \text{ players were obtained in a null round},\\
R'_i &:= \text{ mean number of rounds, starting with } i \text{ players, with failure at the end},\\
\tilde{R}'_i &:= \text{ mean number of rounds, starting with } i \text{ players, with failure at the end},\\
&\quad \text{given that the } i \text{ players were obtained in a null round}.
\end{aligned}$$

Remark: the "$'$" notation will always be used, in the sequel, in relation with the failure case. In case of failure, the moments of $K_n - \log n^*$ are computed as in [10], with some $\tilde{R}'_i$, $\tilde{R}^{(2)\prime}_i$, computed as follows. First we have

$$\begin{aligned}
P'(i) = 1-P(i) &= (p^i)^\tau + \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P'(\ell)\\
&= (p^i)^\tau + \Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P'(\ell),\ i\geq 2,\ P'(1)=0.
\end{aligned}$$

Next the recurrences:

$$\begin{aligned}
P'_K(1,0)=0, &\quad \tilde{P}'_K(1,0)=0,\\
P'_K(1,\geq 1)=0, &\quad \tilde{P}'_K(1,\geq 1)=0,\\
P'_K(i,k) &= (p^i)^\tau[\![k=\tau]\!] + \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P'_K(\ell,k-1-s),\ i\geq 2,\\
\tilde{P}'_K(i,k) &= (p^i)^{\tau-1}[\![k=\tau-1]\!] + \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P'_K(\ell,k-1-s),\ i\geq 2.
\end{aligned}$$

Explanation: We can have τ or $\tau-1$ null rounds (all killed) at start, leading to failure.

$$R'_i = (p^i)^\tau\tau + \Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R'_\ell + \Sigma_2(i,\tau)[P'(i)-(p^i)^\tau]/\Sigma_1(i,\tau),\ R'_1=0,$$

$$\tilde{R}'_i = (p^i)^{\tau-1}(\tau-1) + \Sigma_1(i,\tau-1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R'_\ell$$
$$+ \Sigma_2(i,\tau-1)[P'(i)-(p^i)^\tau]/\Sigma_1(i,\tau),\ \tilde{R}'_1 = 0,$$

and similar equations for $R_i^{(2)\prime}, \tilde{R}_i^{(2)\prime}$. To obtain the moments of $K_n - \log n^*$, we plug $\tilde{R}'_i, \tilde{R}_i^{(2)\prime}$ into the moments given in [10], *based only on* $f_1(\eta)$ as given by (4) of [10], with $\tilde{P}'_K(i,k)$ instead of $P(i,k)$, and again $\eta := k - \log n^*$, i.e.,

$$f_1(\eta) = \sum_{k=1}^{\infty}\sum_{i=2}^{\infty}\exp\Big(-\frac{1}{p}e^{Lk}e^{-L\eta}\Big)\frac{e^{-L\eta i}e^{Lki}}{i!}\tilde{P}'_K(i,k).$$

Indeed, a maximal non-empty urn with only 1 ball leads to a success. Note that to each value $I = i \geq 2$ corresponds $\tilde{P}'(i)$. Also $A_0(\chi_l)$ is no more null here and $\tilde{P}'(1) = 0$ by convention. This gives

Theorem 5.2 *Asymptotic distribution and moments of* $K_n - \log n^*$*, failure case.*

$$\mathbb{P}'(K_n = k) \sim f_1(\eta),$$
$$R'_n - \log n^* = \mathbb{E}(K_n - \log n^*) \sim U_1 - MV_1 - \frac{V_2}{L}$$
$$+ \sum_{l\neq 0}\left[B_1(\chi_l) - MA_0(\chi_l) - \frac{A_1(\chi_l)}{L}\right]e^{-2l\pi\mathbf{i}\{\log n\}},$$
$$\mathbb{E}(K_n - \log n^*)^2 \sim U_2 - 2MU_1 - 2\frac{U_4}{L} + M^2V_1 + 2M\frac{V_2}{L} + \frac{V_3+V_4}{L^2}$$
$$+ \sum_{l\neq 0}\Big\{B_2(\chi_l) - 2MB_1(\chi_l) - 2\frac{B_3(\chi_l)}{L} + M^2A_0(\chi_l)$$
$$+ 2M\frac{A_1(\chi_l)}{L} + \frac{A_2(\chi_l)+A_3(\chi_l)}{L^2}\Big\}e^{-2l\pi\mathbf{i}\{\log n\}}.$$

To obtain the moments of $K_n - \log n^*$, *given failure*, we simply divide the moments given in the theorem by F_n.

6 Asymptotic distribution and moments of T_n (null rounds)

6.1 Asymptotic distribution and moments of T_n (null rounds), with success

Notations:

$P_T(i,t) :=$ Probability that, starting with i players, we succeed with t null rounds,

$\tilde{P}_T(i,t) :=$ Probability that, starting with i players, we succeed with t null rounds, given that the i players were obtained in a null round,

$P_T(t) :=$ Probability that, starting with n players, we succeed with t null rounds,

$I_i :=$ average number of null rounds, starting with i players, with success at the end,

$\tilde{I}_i :=$ average number of null rounds, starting with i players, with success at the end, given that the i players were obtained in a null round.

The analysis is similar to that of K_n. We have the following recurrences:

$$P_T(1,0) = 1, \quad \tilde{P}_T(1,0) = 1,$$
$$P_T(1,\geq 1) = 0, \quad \tilde{P}_T(1,\geq 1) = 0,$$

$$P_T(i,t) = \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-s),\ \ i \geq 2,$$
$$\tilde{P}_T(i,t) = \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-1-s),\ \ i \geq 2.$$

Explanation: We can have up to $\tau-1$ or $\tau-2$ null rounds (all killed), followed by ℓ survivors. This leads to s or $s+1$ null rounds already.

$$\begin{aligned}
I_i &= \sum_t P_T(i,t)t = \sum_t \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-s)[t-s+s],\\
&= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_\ell + \sum_t \sum_{s=0}^{\tau-1} (p^i)^s s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-s)\\
&= \Sigma_1(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_\ell + \Sigma_4(i,\tau)P(i)/\Sigma_1(i,\tau),\ \ I_1 = 0,\\
\tilde{I}_i &= \sum_t \tilde{P}_T(i,t)t = \sum_t \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-1-s)[t-1-s+s+1],\\
&= \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_\ell + \sum_t \sum_{s=0}^{\tau-2} (p^i)^s (s+1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-1-s)\\
&= \Sigma_1(i,\tau-1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_\ell + \Sigma_2(i,\tau-1)P(i)/\Sigma_1(i,\tau),\ \ \tilde{I}_1 = 1,\\
I_i^{(2)} &= \sum_t P_T(i,t)t^2 = \sum_t \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-s)[t-s+s]^2,\\
&= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_\ell^{(2)} + \sum_t \sum_{s=0}^{\tau-1} (p^i)^s s^2 \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-1-s)\\
&+ 2\sum_{s=0}^{\tau-1} (p^i)^s s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_\ell\\
&= \Sigma_1(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_\ell^{(2)} + \Sigma_5(i,\tau)P(i)/\Sigma_1(i,\tau) + 2\Sigma_4(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_\ell,\ \ I_1^{(2)} = 1,\\
\tilde{I}_i^{(2)} &= \sum_t \tilde{P}_T(i,t)t^2 = \sum_t \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-1-s)[t-1-s+s+1]^2,\\
&= \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_\ell^{(2)} + \sum_t \sum_{s=0}^{\tau-2} (p^i)^s (s+1)^2 \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-1-s)\\
&+ 2\sum_{s=0}^{\tau-2} (p^i)^s (s+1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_\ell\\
&= \Sigma_1(i,\tau-1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_\ell^{(2)} + \Sigma_3(i,\tau-1)P(i)/\Sigma_1(i,\tau)\\
&+ 2\Sigma_2(i,\tau-1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_\ell,\ \ \tilde{I}_1^{(2)} = 0.
\end{aligned}$$

Again, as in previous expressions, we could replace

$$\sum_{\ell=1}^{i}\binom{i}{\ell}q^{\ell}p^{i-\ell}I_{\ell} \quad \text{by} \quad [I_i-\Sigma_4(i,\tau)P(i)/\Sigma_1(i,\tau)]/\Sigma_1(i,\tau).$$

Next, with (1) and $\eta := j-\log n^*$,

$$\mathbb{P}(J=j,T_n=t)\sim f_3(\eta,t),$$
$$f_3(\eta,t)=\sum_{i=2}^{\infty}\exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{e^{-L\eta i}}{i!}\tilde{P}_T(i,t).$$

Hence

$$\phi_3(\alpha,t)=\int_{-\infty}^{\infty}e^{\alpha\eta}f_3(\eta,t)d\eta=\sum_{i=2}^{\infty}\frac{(1/p)^{-i+\tilde{\alpha}}}{Li!}\Gamma(i-\tilde{\alpha})\tilde{P}_T(i,t).$$

Note that there are no null rounds if the maximal non-empty urn contains only 1 ball.

The dominant component of $P_T(t)$ is given by

$$\phi_3(0,t)=\sum_{i=2}^{\infty}\frac{(1/p)^{-i}}{Li!}\Gamma(i)\tilde{P}_T(i,t)=\sum_{i=2}^{\infty}\frac{p^i}{Li}\tilde{P}_T(i,t),$$

and the periodic component by

$$\omega_{1,3}(t)=\sum_{l\neq 0}\varphi_3(\chi_l,t)e^{-2l\pi\mathbf{i}\{\log n^*\}},$$

with

$$\varphi_3(\chi_l,t)=\phi_3(\alpha,t)\big|_{\alpha=-L\chi_l}.$$

We obtain

$$\varphi_3(\chi_l,t)=\sum_{i=2}^{\infty}\frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{P}_T(i,t).$$

Hence we have the following theorem.

Theorem 6.1 *The asymptotic distribution of the number T_n of null rounds, with success, is given by*

$$P_T(t)=\mathbb{P}(T_n=t)\sim\sum_{i=2}^{\infty}\frac{p^i}{Li}\tilde{P}_T(i,t)+\sum_{l\neq 0}\varphi_3(\chi_l,t)e^{-2l\pi\mathbf{i}\{\log n^*\}}.$$

The moments are given by

$$I_n\sim\sum_{i=2}^{\infty}\frac{p^i}{Li}\tilde{I}_i+\sum_{l\neq 0}\sum_{i=2}^{\infty}\frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{I}_ie^{-2l\pi\mathbf{i}\{\log n^*\}},$$
$$I_n^{(2)}\sim\sum_{i=2}^{\infty}\frac{p^i}{Li}\tilde{I}_i^{(2)}+\sum_{l\neq 0}\sum_{i=2}^{\infty}\frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{I}_i^{(2)}e^{-2l\pi\mathbf{i}\{\log n^*\}}.$$

Note that $T_n=\mathcal{O}(1)$.

6.2 Asymptotic distribution and moments of T_n (null rounds), with failure

Notations:

$$P'_T(i,t) := \text{Probability that, starting with } i \text{ players, we fail with } t \text{ null rounds,}$$
$$\tilde{P}'_T(i,t) := \text{Probability that, starting with } i \text{ players, we fail with } t \text{ null rounds, given that the } i \text{ players were obtained in a null round,}$$
$$P'_T(t) := \text{Probability that, starting with } n \text{ players, we fail with } t \text{ null rounds,}$$
$$I'_i := \text{average number of null rounds, starting with } i \text{ players, with failure at the end,}$$
$$\tilde{I}'_i := \text{average number of null rounds, starting with } i \text{ players, with failure at the end, given that the } i \text{ players were obtained in a null round.}$$

We have the recurrences:

$$P'_T(1,0) = 0, \quad \tilde{P}'_T(1,0) = 0,$$
$$P'_T(1,\geq 1) = 0, \quad \tilde{P}'_T(1,\geq 1) = 0,$$
$$P'_T(i,t) = (p^i)^\tau [\![t=\tau]\!] + \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P'_T(\ell, t-s),$$
$$\tilde{P}'_T(i,t) = (p^i)^{\tau-1} [\![t=\tau]\!] + \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P'_T(\ell, t-1-s).$$

Explanation: We can have τ or $\tau-1$ null rounds (all killed) at start, leading to failure.

$$\begin{aligned}
I'_i &= \sum_t P'_T(i,t)t = (p^i)^\tau\tau + \sum_t \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P'_T(\ell,t-s)[t-s+s] \\
&= (p^i)^\tau\tau + \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} I'_\ell + \sum_t \sum_{s=0}^{\tau-1}(p^i)^s s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P'_T(\ell,t-s) \\
&= (p^i)^\tau\tau + \Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} I'_\ell + \Sigma_4(i,\tau)[P'(i)-(p^i)^\tau]/\Sigma_1(i,\tau), \; I'_1 = 0,
\end{aligned}$$

$$\begin{aligned}
\tilde{I}'_i &= \sum_t \tilde{P}'_T(i,t)t = (p^i)^{\tau-1}\tau + \sum_t \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P'_T(\ell,t-1-s)[t-1-s+s+1] \\
&= (p^i)^{\tau-1}\tau + \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} I'_\ell + \sum_t \sum_{s=0}^{\tau-2}(p^i)^s (s+1) \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P'_T(\ell,t-1-s) \\
&= (p^i)^{\tau-1}\tau + \Sigma_1(i,\tau-1)\sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} I'_\ell + \Sigma_2(i,\tau-1)[P'(i)-(p^i)^\tau]/\Sigma_1(i,\tau), \; \tilde{I}'_1 = 0.
\end{aligned}$$

Next, with (1), again with $\eta := j - \log n^*$,

$$\mathbb{P}'(J=j, T_n=t) \sim f_4(\eta,t),$$
$$f_4(\eta,t) = \sum_{i=2}^{\infty} \exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{e^{-L\eta i}}{i!}\tilde{P}'_T(i,t).$$

Hence

$$\phi_4(\alpha,t) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_4(\eta,t)d\eta = \sum_{i=2}^{\infty}\frac{(1/p)^{-i+\tilde{\alpha}}}{Li!}\Gamma(i-\tilde{\alpha})\tilde{P}'_T(i,t).$$

Note that there are no null rounds if the maximal non-empty urn contains only 1 ball.

The dominant component of $P'_T(t)$ is given by

$$\phi_4(0,t) = \sum_{i=2}^{\infty} \frac{p^i}{Li!}\Gamma(i)\tilde{P}'_T(i,t),$$

and the periodic component by

$$\omega_{1,4}(t) = \sum_{l\neq 0} \varphi_4(\chi_l,t)e^{-2l\pi\mathbf{i}\{\log n^*\}},$$

with

$$\varphi_4(\chi_l,t) = \phi_4(\alpha,t)\big|_{\alpha=-L\chi_l}.$$

We obtain

$$\varphi_4(\chi_l,t) = \sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{P}'_T(i,t).$$

Hence

Theorem 6.2 *The asymptotic distribution of the number T_n of null rounds, with failure, is given by*

$$P'_T(t) = \mathbb{P}'(T_n = t) \sim \sum_{i=2}^{\infty} \frac{p^i}{Li}\tilde{P}'_T(i,t) + \sum_{l\neq 0} \varphi_4(\chi_l,t)e^{-2l\pi\mathbf{i}\{\log n^*\}}.$$

The moments are given by

$$I'_n \sim \sum_{i=2}^{\infty} \frac{p^i}{Li}\tilde{I}_{i,F} + \sum_{l\neq 0}\sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{I}'_i e^{-2l\pi\mathbf{i}\{\log n^*\}},$$

$$I^{(2)\prime}_n \sim \sum_{i=2}^{\infty} \frac{p^i}{Li}\tilde{I}^{(2)}_{i,F} + \sum_{l\neq 0}\sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{I}^{(2)\prime}_i e^{-2l\pi\mathbf{i}\{\log n^*\}}.$$

Note that $T_n = \mathcal{O}(1)$.

7 Asymptotic distribution and moments of W_n (leftovers), with failure

Notations:

$P'_W(i,r)$:= Probability that, starting with i players, we fail, with r players remaining at the end (leftovers),

$\tilde{P}'_W(i,r)$:= Probability that, starting with i players, we fail, with r players remaining at the end (leftovers), given that the i players were obtained in a null round,

$P'_W(r)$:= Probability that, starting with n players, we fail, with r players remaining at the end (leftovers),

L'_i := average number of leftovers, starting with i players, with failure at the end,

$\tilde{L}'_i$:= average number of leftovers, starting with i players, with failure at the end, given that the i players were obtained in a null round.

We have the following recurrences:

$$P'_W(r,r) = (p^r)^\tau + \Sigma_1(r,\tau)q^r P'_W(r,r), \text{ hence}$$

$$P'_W(r,r) = \frac{p^{r\tau}(1-p^r)}{1-p^r-q^r+q^r p^{r\tau}},$$
$$\tilde{P}'_W(r,r) = (p^r)^{\tau-1} + \Sigma_1(r,\tau-1) q^r P'_W(r,r).$$

Explanation: We have r players alive at start or r players alive before the starting null round.

$$P'_W(i,r) = \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=r}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P'_W(\ell,r) = \Sigma_1(i,\tau)\sum_{\ell=r}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P'_W(\ell,r),\ i>r,\ i\geq 2,$$
$$\tilde{P}'_W(i,r) = \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=r}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P'_W(\ell,r) = \Sigma_1(i,\tau-1)\sum_{\ell=r}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P'_W(\ell,r),\ i>r,\ i\geq 2,$$
$$= \Sigma_1(i,\tau-1)/\Sigma_1(i,\tau) P'_W(i,r),\ i>r,\ i\geq 2.$$

Explanation: We can have up to $\tau-1$ or $\tau-2$ null rounds (all killed), followed by ℓ survivors.

$$\begin{aligned}
L'_i &= \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^i p^{i\tau}} i + \sum_{r=0}^{i-1} P'_W(i,r) r \\
&= \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^i p^{i\tau}} i + \Sigma_1(i,\tau)\sum_{r=0}^{i-1}\sum_{\ell=r}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P'_W(\ell,r) r, \\
&= \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^i p^{i\tau}} i + \Sigma_1(i,\tau)\left[\sum_{\ell=0}^{i-1}\binom{i}{\ell} q^\ell p^{i-\ell}\sum_{r=0}^{\ell} P'_W(\ell,r) r + \sum_{r=0}^{i-1} q^i P'_W(i,r) r\right] \\
&= \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^i p^{i\tau}} i + \Sigma_1(i,\tau)\left[\sum_{\ell=0}^{i-1}\binom{i}{\ell} q^\ell p^{i-\ell} L'_\ell + q^i\left[L'_i - \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^i p^{i\tau}} i\right]\right], \\
\tilde{L}'_i &= \frac{\Sigma_1(i,\tau-1)}{\Sigma_1(i,\tau)}\sum_{r=0}^{i-1} P'_W(i,r) r + \left[(p^i)^{\tau-1} + \Sigma_1(i,\tau-1) q^i P'_W(i,i)\right] i,
\end{aligned}$$

and similar equations for $L_i^{(2)\prime}$, $\tilde{L}_i^{(2)\prime}$.

Next, with (1) and $\eta := j - \log n^*$,

$$\mathbb{P}'(J=j, W_n=r) \sim f_5(\eta,r),$$
$$f_5(\eta,r) = \sum_{i=2}^{\infty}\exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{e^{-L\eta i}}{i!}\tilde{P}'_W(i,r).$$

Hence

$$\phi_5(\alpha,r) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_5(\eta,r)d\eta = \sum_{i=2}^{\infty}\frac{(1/p)^{-i+\tilde{\alpha}}}{Li!}\Gamma(i-\tilde{\alpha})\tilde{P}'_W(i,r).$$

Note that there are no leftovers if the maximal non-empty urn contains only 1 ball.

The dominant component of $P'_W(r)$ is given by

$$\phi_5(0,r) = \sum_{i=2}^{\infty}\frac{(1/p)^{-i}}{Li!}\Gamma(i)\tilde{P}'_W(i,r) = \sum_{i=2}^{\infty}\frac{p^i}{Li}\Gamma(i)\tilde{P}'_W(i,r),$$

and the periodic component by

$$\omega_{1,5}(r) = \sum_{l\neq 0}\varphi_5(\chi_l,r)e^{-2l\pi\mathbf{i}\{\log n^*\}},$$

with

$$\varphi_5(\chi_l,r) = \phi_5(\alpha)\big|_{\alpha=-L\chi_l}.$$

We obtain

$$\varphi_5(\chi_l, r) = \sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{P}'_W(i,r).$$

Hence

Theorem 7.1 *The asymptotic distribution of the number W_n of leftovers, with failure, is given by*

$$P'_W(r) = \mathbb{P}'(W_n = r) \sim \sum_{i=2}^{\infty} \frac{p^i}{Li}\frac{\Sigma_1(i,\tau-1)}{\Sigma_1(i,\tau)}P'_W(i,r) + \sum_{l\neq 0}\varphi_5(\chi_l, r)e^{-2l\pi\mathbf{i}\{\log n^*\}}.$$

The moments are given by

$$L'_n \sim \sum_{i=2}^{\infty}\frac{p^i}{Li}\tilde{L}'_i + \sum_{l\neq 0}\sum_{i=2}^{\infty}\frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{L}'_i e^{-2l\pi\mathbf{i}\{\log n^*\}},$$

$$L_n^{(2)\prime} \sim \sum_{i=2}^{\infty}\frac{p^i}{Li}\tilde{L}_i^{(2)\prime} + \sum_{l\neq 0}\sum_{i=2}^{\infty}\frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{L}_i^{(2)\prime} e^{-2l\pi\mathbf{i}\{\log n^*\}}.$$

Note that $W_n = \mathcal{O}(1)$.

8 Asymptotic distribution and moments of C_n (coins flipped)

8.1 Asymptotic distribution and moments of C_n (coins flipped), with success

This RV is more delicate to analyze. In previous cases, all interesting RV were related to urns (at high level) containing $\mathcal{O}(1)$ balls. Here *all* urns contribute to C_n, so we must include the contribution of urns before J, which actually lead to the dominant part of C_n. Also some correlations must be taken into account. We obtain the dominant and corrected terms of the moments as well as a central limit theorem.

Notations:

$N_i :=$ Average number of coins flipped, starting with i players, with success at the end,

$\tilde{N}_i :=$ average number of coins flipped, starting with i players, with success at the end, given that the i players were obtained in a null round.

8.1.1 Case $I = 1$.

Note that, as explained in Section 4, this case *entails a success*. We will only deal here with the non-periodic part of our expressions. The maximal non-empty urn contains 1 ball and the position of the last non-empty urn *before* this maximal non-empty urn is denoted by J. Let us also denote by K the number of balls in urn J.

$$\mathbb{P}(J=j, K=k) \sim f_6(\eta, k),\ \ k \geq 1,$$

$$f_6(\eta,k) := \exp\Big(-\frac{q}{p}e^{-L\eta}\Big)\frac{q}{p}e^{-L\eta}\exp\left(-e^{-L\eta}\right)\frac{e^{-L\eta k}}{k!},$$

$$= \exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{q}{p}\frac{e^{-L\eta(k+1)}}{k!}.$$

Explanation:

$$\mathbb{P}(J=j, K=k) \sim \mathcal{P}\Big(\frac{q}{p}e^{-L\eta}, 1\Big)\mathcal{P}\left(e^{-L\eta}, k\right).$$

We have

$$\phi_6(\alpha, k) = \int_{-\infty}^{\infty} e^{\alpha\eta}f_6(\eta,k)d\eta = \frac{q}{Lk!}\left(\frac{1}{p}\right)^{\tilde{\alpha}-k}\Gamma(1-\tilde{\alpha}+k),$$

$$\Pi_4(k) := \phi_6(0,k) = \frac{q}{L}p^k.$$

Note that

$$Z_1 := \sum_{k=1}^{\infty} \Pi_4(k) = \frac{p}{L} \equiv \Pi_1 \quad \text{(one ball in the maximal non-empty urn)}$$

which conforms to (3).

Let us denote by Δ the *difference* between the maximal non-empty urn (containing 1 ball) and J. We have

$$\begin{aligned}
\mathbb{P}(J=j, I=1, \Delta=\delta) &\sim f_7(\eta,\delta),\\
f_7(\eta,\delta) &:= \exp\Big(-\frac{q}{p}e^{-L\eta}\Big)e^{-L(\eta+\delta)}\left(1-\exp\left(-e^{-L\eta}\right)\right),\\
&= \exp(-L\delta)\exp\Big(-\frac{q}{p}e^{-L\eta}\Big)e^{-L\eta}\left(1-\exp\left(-e^{-L\eta}\right)\right),
\end{aligned}$$

which shows that Δ is asymptotically independent of J.

Explanation:

$$\begin{aligned}
\mathbb{P}(J=j, I=1, \Delta=\delta) &\sim \mathcal{P}\left(e^{-L(\eta+1)},0\right)\mathcal{P}\left(e^{-L(\eta+2)},0\right)\dots\\
&\dots\mathcal{P}\left(e^{-L(\eta+\delta)},1\right)\mathcal{P}\left(e^{-L(\eta+\delta+1)},0\right)\dots\left[1-\mathcal{P}\left(e^{-L\eta},0\right)\right].
\end{aligned}$$

We have

$$\begin{aligned}
\phi_7(\alpha,\delta) &= \int_{-\infty}^{\infty} e^{\alpha\eta} f_7(\eta,\delta)d\eta\\
&= \exp(-L\delta)\frac{p}{Lq}\left[\left(\frac{q}{p}\right)^{\tilde{\alpha}} - q\left(\frac{1}{p}\right)^{\tilde{\alpha}}\right]\Gamma(1-\tilde{\alpha}),\\
\Pi_5(\delta) &:= \phi_7(0,\delta) = e^{-L\delta}\frac{p^2}{Lq} = q^\delta\frac{p^2}{Lq}.
\end{aligned}$$

Note that

$$\sum_{\delta=1}^{\infty} \Pi_5(\delta) = \frac{p}{L} \equiv \Pi_1$$

which again conforms to (3). We have

$$\mathbb{E}(\Delta) = \frac{1}{L} \quad \text{and} \quad \mathbb{E}(\Delta^2) = \frac{1+q}{Lp}.$$

However, note carefully that the player corresponding to $I=1$ is actually related to a *flipped coin in urn* J. So we must use a new RV G, denoting the number of flipped coins at step J: $G = K+1$, $G \geq 2$, with distribution

$$\Pi_6(g) = \frac{q}{L}p^{g-1},\ g \geq 2$$

and

$$f_8(\eta,g) = \exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{q}{p}\frac{e^{-L\eta g}}{(g-1)!}.$$

We will also need

$$Z_5 = \mathbb{E}(G) := \sum_{g=2}^{\infty} \Pi_6(g)g = \frac{p(1+q)}{Lq}.$$

Later on, we will use the following variants:

$$e^{-L\eta}f_8(\eta,g),\quad e^{-2L\eta}f_8(\eta,g),\quad \eta f_8(\eta,g),\quad e^{-L\eta}\eta f_8(\eta,g),\quad e^{-2L\eta}\eta f_8(\eta,g).$$

These variants lead respectively to $\phi_{.}(0,g)$:

$$
\begin{aligned}
&\frac{p^g qg}{L},\\
&\frac{qp^{g+1}g(g+1)}{L},\\
&-\frac{p^{g-1}q[(g-1)\ln(p)+(g-1)\psi(g-1)+1]}{L^2(g-1)},\\
&-\left[qp^g[2(g-1)+(g-1)^2\ln(p)+(g-1)^2\psi(g-1)+(g-1)\ln(p)\right.\\
&\left.+(g-1)\psi((g-1))+1]\right]/[L^2(g-1)],\\
&\Omega_{15}(g) \quad \text{is too long to be displayed here.}
\end{aligned}
$$

This leads to $Z_7, Z_8, Z_{11}, Z_{12}, Z_{10}, Z_{13}, Z_{15}$ as given in Appendix A: we *simply sum on* $g \geq 2$. Indeed, the case $I = 1$ immediately leads to a success.

Now we will separate the contribution of urn J (containing G balls) from that of urns $< J$.

Let us denote by $S_\Gamma(j,i)$ the sum of $(n-i)$ iid RV $\Gamma(j)$, and $\Gamma(j)$ is a truncated geometric RV $< j$. As $\Sigma_0 := \sum_{l=1}^{j-1} pq^{l-1} = 1 - q^{j-1}$, we have (we give only the terms needed in the sequel)

$$
\begin{aligned}
E(j) &:= \mathbb{E}(\Gamma(j)) = \sum_{l=1}^{j-1} pq^{l-1} l/\Sigma_0 \sim \frac{1}{p} + q^{j-1} - jq^{j-1} - jq^{2(j-1)} + \mathcal{O}\big(q^{2(j-1)}\big),\\
E^{(2)}(j) &:= \mathbb{E}(\Gamma(j)^2) = \sum_{l=1}^{j-1} pq^{l-1} l^2/\Sigma_0 \sim \frac{1+q}{p^2} + q^{j-1}\frac{1+q}{p} - j\frac{2q}{p}q^{j-1}\\
&\quad - j^2 q^{j-1} + \mathcal{O}\big(j^2 q^{2(j-1)}\big).
\end{aligned}
$$

Note that, with $j = \eta + \log n^*$,

$$
q^j = e^{-L\eta}\frac{1}{n^*}.
$$

This leads, by carefully taking into account the *correlation* between J and G (we expand the mean up to the $\log n^*/n^*$ term and the square mean up to the $\log n^*$ term) to (there are $n-G$ dead players before attaining step J)

$$
C_{n,1} \sim S_\Gamma(J,G) + JG, \tag{4}
$$

$$
\begin{aligned}
\mathbb{E}(C_{n,1}) &\sim \mathbb{E}(S_\Gamma(J,G) + JG),\\
&\sim \mathbb{E}\Big[(n-G)\frac{1}{p} + \frac{n}{q}\frac{e^{-L\eta}}{n^*} - \frac{n-G}{q}\frac{e^{-L\eta}}{n^*}(\log n^* + \eta)\\
&\quad - \frac{n}{q^2}\frac{e^{-2L\eta}}{n^{*2}}\log n^* + (\log n^* + \eta)G\Big]\\
&\sim \frac{n}{p}Z_1 - \frac{1}{p}Z_5 + \frac{n}{q}\frac{Z_7}{n^*} - \frac{n}{q}\frac{Z_7}{n^*}\log n^* + \frac{1}{q}\frac{Z_8}{n^*}\log n^*\\
&\quad - \frac{n}{q}\frac{Z_{10}}{n^*} - \frac{n}{q^2}\frac{Z_{11}}{n^{*2}}\log n^* + Z_5\log n^* + Z_{13},
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\mathbb{E}(C_{n,1}^2) &\sim \mathbb{E}((S_\Gamma(J,G) + JG)^2)\\
&\sim \mathbb{E}[n\mathbb{E}^{(2)}(J) + (n-G)(n-G-1)(\mathbb{E}(J))^2\\
&\quad + 2\mathbb{E}[(n-G)E(J)JG] + \mathbb{E}[(\log n^* + \eta)^2 G^2].
\end{aligned} \tag{6}
$$

8.1.2 Case $I > 1$.

First of all, we must compute the moments of C_i and $\tilde{C}_i$. This gives

$$\begin{aligned}
N_i &= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell}[si + i + N_\ell], \ N_1 = 0, \\
N_i^{(2)} &= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} \mathbb{E}[((s+1)i + C_\ell)^2] \\
&= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell}[((s+1)i)^2 + 2(s+1)iN_\ell + N_\ell^{(2)}],
\end{aligned}$$

and similar expressions for $\tilde{N}_i$, $\tilde{N}_i^{(2)}$.

Next, with (1),

$$\begin{aligned}
\mathbb{P}(J = j, I = i) &\sim f_9(\eta, i), \\
f_9(\eta, i) &:= \exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{e^{-L\eta i}}{i!}, \\
\mathbb{P}(J = j) &\sim f_{10}(\eta), \\
f_{10}(\eta) &= \sum_{i=2}^{\infty} \exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{e^{-L\eta i}}{i!} = \exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\left(\exp\left(-e^{-L\eta}\right) - 1 - e^{-L\eta}\right), \\
\phi_9(\alpha, i) &= \int_{-\infty}^{\infty} e^{\alpha\eta} f_9(\eta, i) d\eta = \frac{(1/p)^{-i+\tilde{\alpha}}}{Li!}\Gamma(i - \tilde{\alpha}), \\
\Pi_2(i) &:= \phi_9(0, i) = \frac{p^i}{Li}, \\
P_0 &:= \sum_{i=2}^{\infty} \Pi_2(i) = 1 - p/L.
\end{aligned}$$

Later on, we will use the following variants:

$$e^{-L\eta} f_9(\eta, i), \quad e^{-2L\eta} f_9(\eta, i), \quad \eta f_9(\eta, i), \quad e^{-L\eta}\eta f_9(\eta, i), \quad e^{-2L\eta}\eta f_9(\eta, i).$$

These variants lead respectively to $\phi_{.}(0, i)$:

$$\begin{aligned}
&\frac{p^i p}{L}, \\
&\frac{p^i p^2(i+1)}{L}, \\
&-\frac{p^i[\ln(p) + \psi(i)]}{L^2 i}, \\
&-\frac{p^i p[i\ln(p) + i\psi(i) + 1]}{L^2 i}, \\
&-\frac{p^i p^2[i^2\ln(p) + i^2\psi(i) + 2i + i\ln(p) + i\psi(i) + 1]}{L^2 i}.
\end{aligned}$$

Multiplying by $\tilde{P}(i)$ and summing on $i \geq 2$, this leads to $V_7, V_5, V_8, V_{11}, V_{12}, V_{10}, V_{13}, V_{15}$. Indeed, the case $I > 1$ does *not* immediately lead to a success. Again we expand the mean up to the $\log n^*/n^*$ term and the square mean up to the $\log n^*$ term. We have (there are $n - I$ dead players before attaining step J)

$$C_{n,2} \sim S_\Gamma(J, I) + JI + \tilde{C}_I, \tag{7}$$

$$
\begin{aligned}
\mathbb{E}(C_{n,2}) &\sim \mathbb{E}(S_\Gamma(J,I)+JI+\tilde{N}_i)\\
&\sim \mathbb{E}(S_\Gamma(J,I)+JI)+U_1\\
&\sim \mathbb{E}\Big[(n-I)\frac{1}{p}+\frac{n}{q}\frac{e^{-L\eta}}{n^*}-\frac{n-I}{q}\frac{e^{-L\eta}}{n^*}(\log n^*+\eta)\\
&\quad -\frac{n}{q^2}\frac{e^{-2L\eta}}{n^{*2}}\log n^*+(\log n^*+\eta)I\Big]+U_1\\
&\sim \frac{n}{p}V_1-\frac{1}{p}V_5+\frac{n}{q}\frac{V_7}{n^*}-\frac{n}{q}\frac{V_7}{n^*}\log n^*+\frac{1}{q}\frac{V_8}{n^*}\log n^*\\
&\quad -\frac{n}{q}\frac{V_{10}}{n^*}-\frac{n}{q^2}\frac{V_{11}}{n^{*2}}\log n^*+V_5\log n^*+V_{13}+U_1,
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\mathbb{E}(C_{n,2}^2) &\sim \mathbb{E}((S_\Gamma(J,I)+JI)^2)+2\mathbb{E}[(S_\Gamma(J,I)+JI)\tilde{C}_I]\\
&\sim \mathbb{E}((S_\Gamma(J,I)+JI)^2)+2\mathbb{E}\left[\left[\frac{n}{p}-\frac{n}{q}\frac{e^{-L\eta}}{n^*}\log n^*+I\log n^*\right]\tilde{N}_i\right]\\
&\sim \mathbb{E}((S_\Gamma(J,I)+JI)^2)+2\left[\frac{n}{p}U_1-\frac{n}{q}\frac{U_3}{n^*}\log n^*+U_5\log n^*\right]\\
&\sim \mathbb{E}[n\mathbb{E}^{(2)}(J)+(n-I)(n-I-1)(\mathbb{E}(J))^2+2\mathbb{E}[(n-I)E(J)JI]\\
&\quad +\mathbb{E}[(\log n^*+\eta)^2I^2]+2\left[\frac{n}{p}U_1-\frac{n}{q}\frac{U_3}{n^*}\log n^*+U_5\log n^*\right].
\end{aligned}
$$

8.1.3 General case.

The *total mean* is given by (we provide here only two terms)

$$
\begin{aligned}
\mathbb{E}(C_n) &\sim \mathbb{E}(C_{n,1})+\mathbb{E}(C_{n,2})\\
&\sim n\left(\frac{p}{L}+V_1\right)\frac{1}{p}+\left(-\frac{Z_7}{p}+Z_5-\frac{V_7}{p}+V_5\right)\log n^*.
\end{aligned}
$$

Recall that $\frac{p}{L}+V_1$ is equal to $P_d(S)$ as defined before. But the first term amounts to the mean of a sum of n GEOM pq^{l-1} RVs. (Indeed, the GEOM pq^{l-1} RV has mean $\frac{1}{p}$, second moment $\frac{1+q}{p^2}$ and variance $\frac{q}{p^2}$). This is easy to explain: from (4) and (7), the correction $\tilde{C}_{I,S}$ is asymptotically $\mathcal{O}(1)$ and the correction $-\Delta$ is also asymptotically $\mathcal{O}(1)$. Similarly

$$
\begin{aligned}
\mathbb{E}(C_n^2) &\sim \mathbb{E}(C_{n,1}^2)+\mathbb{E}(C_{n,2}^2)\\
&\sim n^2\left(\frac{p}{L}+V_1\right)\frac{1}{p^2}+n\left(-\frac{2(Z_7-pZ_5)}{p^2}-\frac{2(V_7-pV_5)}{p^2}\right)\log n^*
\end{aligned}
$$

and the *variance* is finally given by (we must adequately condition on the dominant success probability $Pd(S):=\frac{p}{L}+V_1$)

$$
\mathbb{V}(C_n)\sim Pd(S)\left[\frac{\mathbb{E}(C_n^2)}{Pd(S)}-\left(\frac{\mathbb{E}(C_n)}{Pd(S)}\right)^2\right]\sim Pd(S)n\frac{q}{p^2}.
$$

So we obtain

Theorem 8.1 *The moments of C_n in case of success, are given by (with Maple, more terms could be provided, in particular the $\log^2 n^*$ and $\log n^*$ terms of the variance)*

$$
\begin{aligned}
\mathbb{E}(C_n) &\sim n\left(\frac{p}{L}+V_1\right)\frac{1}{p}+\left(-\frac{Z_7}{p}+Z_5-\frac{V_7}{p}+V_5\right)\log n^*,\\
\mathbb{V}(C_n) &\sim Pd(S)n\frac{q}{p^2}.
\end{aligned}
$$

Note again that the dominant term of the variance corresponds to a *sum of* n *iid GEOM* pq^{l-1} *RVs*. Intuitively, the asymptotic distribution should be Gaussian: again from (4) and (7), the correction $\tilde{C}_{I,S}$ is asymptotically $\mathcal{O}(1)$, but *not independent* of the dominant term and the correction $-\Delta$ is also asymptotically $\mathcal{O}(1)$, but *independent* of the dominant term. Actually we have the following theorem.

Theorem 8.2 *Conditioned on a success,*

$$\mathbb{P}\left[\frac{C_n - \mathbb{E}(C_n)}{\sqrt{\mathbb{V}(C_n)}} \le x\right] \underset{n\to\infty}{\longrightarrow} \phi(x),$$

where $\phi(x)$ *denotes the Gaussian distribution function.*

The proof is given in Appendix H.

See also Kalpathy et al. [5], for a leader election scheme which stops if $I > 1$. In this model, C_n is shown to be asymptotically Gaussian.

8.2 Distribution of C_n (number coins flipped), with failure

Only the case $I > 1$ matters here. Proceeding as before (we omit the details), we finally derive

Theorem 8.3 *The moments of* C_n *in case of failure, are given by*

$$\mathbb{E}(C_n) \sim nV_1\frac{1}{p} + \Big(-\frac{V_7}{p} + V_5\Big)\log n^*,$$
$$\mathbb{V}(C_n) \sim V_1 n\frac{q}{p^2}.$$

Again, the distribution should be asymptotically Gaussian, but we did not check the details.

Acknowledgments

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Appendices.

A Some definitions and identities

Here, $\xi \in \{R, I, L, N\}$. ξ_i must be replaced by ξ_i or ξ_i' depending on the case we consider. Also $\tilde{P}(i)$ must be replaced by $\tilde{P}(i)$ or $\tilde{P}'(i)$, respectively. Note that, compared with [10], we use here $\Pi_2(i) = \frac{p^i}{Li}$ instead of $\frac{p^i}{i}$, for $V_1, \dots, V_5$. Also we have $\tilde{P}(1) = 1, \tilde{P}'(1) = 0$.

$$V_1 := \sum_{i=2}^{\infty} \frac{p^i}{Li}\tilde{P}(i), \quad V_2 := \sum_{i=2}^{\infty} \frac{p^i\psi(i)}{Li}\tilde{P}(i), \quad V_3 := \sum_{i=2}^{\infty} \frac{p^i\psi(i)^2}{Li}\tilde{P}(i), \quad V_4 := \sum_{i=2}^{\infty} \frac{p^i\psi(1,i)}{Li}\tilde{P}(i), \quad V_5 := \sum_{i=2}^{\infty} \frac{p^i i}{Li}\tilde{P}(i),$$

$$V_7 := \sum_{i=2}^{\infty} \frac{p^i p}{L}\tilde{P}(i) = pV_1, \quad V_8 := \sum_{i=2}^{\infty} \frac{p^i pi}{L}\tilde{P}(i) = pV_5, \quad V_{10} := \sum_{i=2}^{\infty} -\frac{p^i p[i\ln(p) + i\psi(i) + 1]}{L^2 i}\tilde{P}(i),$$

$$V_{11} := \sum_{i=2}^{\infty} \frac{p^i p^2(i+1)}{L}\tilde{P}(i) = p^2V_5 + p^2V_1, \quad V_{12} := \sum_{i=2}^{\infty} -\frac{p^i[\ln(p) + \psi(i)]}{L^2 i}\tilde{P}(i), \quad V_{13} := \sum_{i=2}^{\infty} -\frac{p^i[\ln(p) + \psi(i)]i}{L^2 i}\tilde{P}(i),$$

$$V_{15} := \sum_{i=2}^{\infty} -\frac{p^i p^2[-i^2\ln(p) - i^2\psi(i) - 2i + i\ln(p) - i\psi(i) + 1]}{L^2 i}\tilde{P}(i), \quad V_{16} := \sum_{i=2}^{\infty} -\frac{p^i p[i\ln(p) + i\psi(i) + 1]i}{L^2 i}\tilde{P}(i),$$

$$A_0(\chi_l) := \sum_{i=1}^{\infty} \frac{p^i}{Li!}\Gamma(i+\chi_l)\tilde{P}(i), \quad A_1(\chi_l) := \sum_{i=1}^{\infty} \frac{p^i}{Li!}\Gamma(i+\chi_l)\psi(i+\chi_l)\tilde{P}(i),$$

$$A_2(\chi_l) := \sum_{i=1}^{\infty} \frac{p^i}{Li!}\Gamma(i+\chi_l)\psi(1,i+\chi_l)\tilde{P}(i), \quad A_3(\chi_l) := \sum_{i=1}^{\infty} \frac{p^i}{Li!}\Gamma(i+\chi_l)\psi^2(i+\chi_l)\tilde{P}(i),$$

$$B_1(\chi_l) := \sum_{i=2}^{\infty} \frac{p^i}{Li!}\tilde{\xi}_i\Gamma(i+\chi_l), \quad B_2(\chi_l) := \sum_{i=2}^{\infty} \frac{p^i}{Li!}\tilde{\xi}_i^{(2)}\Gamma(i+\chi_l), \quad B_3(\chi_l) := \sum_{i=2}^{\infty} \frac{p^i}{Li!}\tilde{\xi}_i\psi(i+\chi_l)\Gamma(i+\chi_l),$$

$$U_1 := \sum_{i=2}^{\infty} \frac{p^i\tilde{\xi}_i}{Li}, \quad U_2 := \sum_{i=2}^{\infty} \frac{p^i\tilde{\xi}_i^{(2)}}{Li}, \quad U_3 := \sum_{i=2}^{\infty} \frac{pp^i\tilde{\xi}_i}{L}, \quad U_4 := \sum_{i=2}^{\infty} \frac{p^i\tilde{\xi}_i\psi(i)}{i}, \quad U_5 := \sum_{i=2}^{\infty} \frac{p^i i\tilde{\xi}_i}{Li},$$

$$Z_7 := \frac{p^2[2-3p+p^2]}{q^2L}, \quad Z_8 := \sum_{g=2}^{\infty} \frac{p^g q g^2}{L} = \frac{p^2[4-3p+p^2]}{q^2L}, \quad Z_{11} := \frac{2p^3(p^2-3p+3)}{Lq^2},$$

$$Z_{12} := \sum_{g=2}^{\infty} -\frac{p^{g-1}q[(g-1)\ln(p)+(g-1)\psi(g-1)+1]}{L^2(g-1)},$$

$$Z_{10} := \sum_{g=2}^{\infty} -\frac{qp^g[2(g-1)+(g-1)^2\ln(p)+(g-1)^2\psi((g-1))+(g-1)\ln(p)+(g-1)\psi((g-1))+1]}{L^2(g-1)},$$

$$Z_{13} := \sum_{g=2}^{\infty} -\frac{p^{g-1}q[(g-1)\ln(p)+(g-1)\psi((g-1))+1]g}{L^2(g-1)}, \quad Z_{15} := \sum_{g=2}^{\infty} \Omega_{15}(g).$$

B Success Probability

We show here that, where the results are given both, here, and in [7], they coincide. First, we look at Theorem 4.1. The constant is given by

$$\frac{1}{L}\sum_{k\geq 2} S_k \frac{p^k}{k}\frac{1-p^{(\tau-1)k}}{1-p^{\tau k}} + \frac{p}{L},$$

and it should coincide with

$$\frac{1}{L}\left(qp^\tau + \sum_{k\geq 1}\frac{S_k}{k}\left(p^k - \frac{q^k p^{\tau k}}{(1-q^{\tau+1})^k}\right)\right).$$

We use the notation S_n from [7] as in this paper. We have the recursion

$$\frac{1-p^k}{1-p^{\tau k}}S_k = \sum_{j=1}^{k}\binom{k}{j}p^{k-j}q^jS_j.$$

Therefore

$$\begin{aligned}
\sum_{k\geq 2} S_k \frac{p^k}{k}\frac{1-p^{(\tau-1)k}}{1-p^{\tau k}} &= \sum_{k\geq 2} S_k \frac{p^k}{k}\frac{1-p^{\tau k}+p^{\tau k}-p^{(\tau-1)k}}{1-p^{\tau k}} \\
&= \sum_{k\geq 2} S_k\frac{p^k}{k} - \sum_{k\geq 2} S_k\frac{p^{\tau k}}{k}\frac{1-p^k}{1-p^{\tau k}} \\
&= \sum_{k\geq 2} S_k\frac{p^k}{k} - \sum_{k\geq 2}\frac{p^{\tau k}}{k}\sum_{j=1}^{k}\binom{k}{j}p^{k-j}q^jS_j \\
&= \sum_{k\geq 1} S_k\frac{p^k}{k} - p + p^\tau q - \sum_{j\geq 1}\frac{q^jS_j}{j}\sum_{k\geq j}p^{\tau k}\binom{k-1}{j-1}p^{k-j} \\
&= \sum_{k\geq 1} S_k\frac{p^k}{k} - p + p^\tau q - \sum_{j\geq 1}\frac{q^jS_j}{j}\frac{p^{\tau j}}{(1-p^{\tau+1})^j},
\end{aligned}$$

which is the desired formula after trivial modifications.

For the Fourier coefficients, we have to prove that

$$p\Gamma(\chi_l+1) + \sum_{k\geq 2} S_k\frac{p^k}{k!}\frac{1-p^{(\tau-1)k}}{1-p^{\tau k}}\Gamma(\chi_l+k) = qp^\tau\Gamma(\chi_l+1) + \sum_{k\geq 1}\frac{S_k}{k!}\Gamma(\chi_l+k)\left(p^k - \frac{q^kp^{\tau k}}{(1-q^{\tau+1})^{\chi_l+k}}\right),$$

which is done in a similar way:

$$\sum_{k\ge2}S_k\frac{p^k}{k!}\frac{1-p^{(\tau-1)k}}{1-p^{\tau k}}\Gamma(\chi_l+k)=\sum_{k\ge2}S_k\frac{p^k}{k!}\frac{1-p^{\tau k}+p^{\tau k}-p^{(\tau-1)k}}{1-p^{\tau k}}\Gamma(\chi_l+k)$$
$$=\sum_{k\ge1}S_k\frac{p^k}{k!}\Gamma(\chi_l+k)-p\Gamma(\chi_l+1)-\sum_{k\ge2}\frac{p^{\tau k}}{k!}\Gamma(\chi_l+k)\sum_{j=1}^{k}\binom{k}{j}p^{k-j}q^jS_j$$
$$=\sum_{k\ge1}S_k\frac{p^k}{k!}\Gamma(\chi_l+k)-p\Gamma(\chi_l+1)-\sum_{j\ge1}\frac{q^jS_j}{j!}\sum_{k\ge j}p^{\tau k}\Gamma(\chi_l+k)\frac{1}{(k-j)!}p^{k-j}+qp^\tau\Gamma(\chi_l+1)$$
$$=\sum_{k\ge1}S_k\frac{p^k}{k!}\Gamma(\chi_l+k)-p\Gamma(\chi_l+1)+qp^\tau\Gamma(\chi_l+1)-\sum_{j\ge1}\frac{q^jS_j}{j!}\frac{\Gamma(\chi_l+j)p^{\tau j}}{(1-p^{\tau+1})^{\chi_j+k}},$$

which is the formula.

C Total number of rounds

Notations:

$\mathbf{P}_K(i,k) :=$ Probability that, starting with i players, we end after k rounds,
$\tilde{\mathbf{P}}_K(i,k) :=$ Probability that, starting with i players, we end after k rounds,
given that the i players were obtained in a null round, not preceded by another null round.

Next we turn to the nonfluctuating part of R_n. We must use the total number of rounds: $\tilde{\mathbf{R}}_i = \tilde{R}_i + \tilde{R}'_i$. Now from Theorems 5.1 and 5.2, we have

$$\mathbf{R}_n-\log n^*=\mathbb{E}(K_n-\log n^*)\sim\bar{U}_1-M\bar{V}_1-\frac{\bar{V}_2}{L}+\frac{p\gamma}{L^2}-\frac{1+pM}{L}$$
$$+\sum_{l\ne0}\left[\bar{B}_1(\chi_l)-M\bar{A}_0(\chi_l)-\frac{\bar{A}_1(\chi_l)}{L}-\frac{\Gamma(1+\chi_l)}{L}\right]e^{-2l\pi\mathbf{i}\{\log n\}}$$

where, now,

$$\bar{B}_1(\chi_l):=\sum_{i=2}^{\infty}\frac{p^i}{Li!}\tilde{\mathbf{R}}_i\Gamma(i+\chi_l),$$
$$\bar{U}_1:=\sum_{i=2}^{\infty}\frac{p^i\tilde{\mathbf{R}}_i}{Li},$$
$$\bar{A}_0(\chi_l):=0,$$
$$\bar{A}_1(\chi_l):=\Gamma(\chi_l),$$
$$\bar{V}_1:=\frac{L-p}{L},$$
$$\bar{V}_2:=\frac{L}{2}-\frac{\gamma(L-p)}{L}.$$

We have the following recurrences, with $\mathbf{P}_K$ now combining S and F,

$$\mathbf{P}_K(1,0)=1,\tilde{\mathbf{P}}_K(1,0)=1,$$
$$\mathbf{P}_K(i,k)=(p^i)^\tau[\![k=\tau]\!]+\sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{P}_K(\ell,k-1-s),\ i\ge2,$$

$$\tilde{\mathbf{P}}_K(i,k) = (p^i)^{\tau-1}[\![k=\tau-1]\!] + \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell}\mathbf{P}_K(\ell, k-1-s),\ i \geq 2.$$

$$\begin{aligned}
\mathbf{R}_i &= \sum_k \mathbf{P}_K(i,k)k = (p^i)^\tau \tau + \sum_k \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell}\mathbf{P}_K(\ell, k-1-s)[k-1-s+s+1], \\
&= (p^i)^\tau \tau + \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell}\mathbf{R}_\ell + \sum_k \sum_{s=0}^{\tau-1}(p^i)^s(s+1) \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell}\mathbf{P}_K(\ell, k-1-s) \\
&= (p^i)^\tau \tau + \Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell}\mathbf{R}_\ell + \Sigma_2(i,\tau)\left[1-(p^i)^\tau\right]/\Sigma_1(i,\tau), \\
&= \frac{1-p^{i\tau}}{1-p^i}\sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell}\mathbf{R}_\ell + \frac{1-p^{i\tau}}{1-p^i}, \quad \mathbf{R}_1 = 0,
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{R}}_i &= \sum_k \tilde{\mathbf{P}}_K(i,k)k = (p^i)^{\tau-1}(\tau-1) + \Sigma_1(i,\tau-1)\sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell}\mathbf{R}_\ell, \\
&+ \Sigma_2(i,\tau-1)\left[1-(p^i)^\tau\right]/\Sigma_1(i,\tau) \\
&= (p^i)^{\tau-1}(\tau-1) + \Sigma_1(i,\tau-1)\left[\mathbf{R}_i - \Sigma_2(i,\tau)\left[1-(p^i)^\tau\right]/\Sigma_1(i,\tau) - (p^i)^\tau\right]/\Sigma_1(i,\tau) \\
&+ \Sigma_2(i,\tau-1)\left[1-(p^i)^\tau\right]/\Sigma_1(i,\tau) \\
&= \mathbf{R}_i\frac{1-p^i-p^{i(\tau-1)}+p^{i\tau}}{(1-p^{i\tau})(1-p^i)} \\
&= \mathbf{R}_i\frac{1-p^{i(\tau-1)}}{1-p^{i\tau}}, \quad \tilde{\mathbf{R}}_1 = 0.
\end{aligned}$$

In [7], the following recursion is derived:

$$\mathbf{R}_n(\tau) = \sum_{j=1}^{n}\binom{n}{j}p^{n-j}q^j\mathbf{R}_j(\tau) + p^n\mathbf{R}_n(\tau-1) + 1, \quad \tau > 0,\ n \geq 2,$$

and the interest is in $\mathbf{R}_n = \mathbf{R}_n(\tau)$. We write

$$\mathbf{R}_n(\tau) = D_n(\tau) + p^n\mathbf{R}_n(\tau-1) = D_n(\tau) + p^n(K_n(\tau) + \mathbf{R}_n(\tau-2)) = \cdots = \frac{1-p^{\tau n}}{1-p^n}D_n(\tau).$$

We find the recursion

$$\mathbf{R}_n = \frac{1-p^{\tau n}}{1-p^n}\sum_{j=1}^{n}\binom{n}{j}p^{n-j}q^j\mathbf{R}_j + \frac{1-p^{\tau n}}{1-p^n}.$$

This coincides with the recursion given here. Now

$$\begin{aligned}
&\bar{U}_1 - M\bar{V}_1 - \frac{\bar{V}_2}{L} + \frac{p\gamma}{L^2} - \frac{1+pM}{L} \\
&\quad= \sum_{i=2}^{\infty}\frac{p^i\tilde{\mathbf{R}}_i}{Li} - \frac{\ln p}{L}\frac{L-p}{L} - \frac{1}{L}\Big(\frac{L}{2} - \frac{\gamma(L-p)}{L}\Big) + \frac{p\gamma}{L^2} - \frac{1}{L} - \frac{p\ln p}{L^2} \\
&\quad= \frac{1}{L}\sum_{i=2}^{\infty}\frac{p^i(1-p^{i(\tau-1)})}{i}\frac{\mathbf{R}_i}{1-p^{i\tau}} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L} \\
&\quad= \frac{1}{L}\sum_{i=2}^{\infty}\frac{p^i(1-p^{i\tau}+p^{i\tau}-p^{i(\tau-1)})}{i}\frac{\mathbf{R}_i}{1-p^{i\tau}} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L}
\end{aligned}$$

$$= \frac{1}{L}\sum_{i=2}^{\infty}\frac{p^i\mathbf{R}_i}{i} - \frac{1}{L}\sum_{i=2}^{\infty}\frac{p^{i\tau}}{i}\frac{\mathbf{R}_i(1-p^i)}{1-p^{i\tau}} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L}$$

$$= \frac{1}{L}\sum_{i=2}^{\infty}\frac{p^i\mathbf{R}_i}{i} - \frac{1}{L}\sum_{i=1}^{\infty}\frac{p^{i\tau}}{i}\sum_{j=1}^{i}\binom{i}{j}p^{i-j}q^j\mathbf{R}_j - \frac{1}{L}\sum_{i=1}^{\infty}\frac{p^{i\tau}}{i} + \frac{p^\tau}{L} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L}$$

$$= \frac{1}{L}\sum_{i=2}^{\infty}\frac{p^i\mathbf{R}_i}{i} - \frac{1}{L}\sum_{j\geq 1}\frac{q^j\mathbf{R}_j}{j}\sum_{i\geq j}p^{i\tau}\binom{i-1}{j-1}p^{i-j} + \frac{1}{L}\ln(1-p^\tau) + \frac{p^\tau}{L} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L}$$

$$= \frac{1}{L}\sum_{i=2}^{\infty}\frac{p^i\mathbf{R}_i}{i} - \frac{1}{L}\sum_{j\geq 1}\frac{q^j\mathbf{R}_j}{j}\frac{p^{j\tau}}{(1-p^{\tau+1})^j} + \frac{1}{L}\ln(1-p^\tau) + \frac{p^\tau}{L} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L}.$$

Thus the constant term in the asymptotic expansion of $\mathbf{R}_n - \log n$ is

$$\frac{\ln p}{L} + 1 + \frac{1}{L}\sum_{i=2}^{\infty}\frac{p^i\mathbf{R}_i}{i} - \frac{1}{L}\sum_{j\geq 1}\frac{q^j\mathbf{R}_j}{j}\frac{p^{j\tau}}{(1-p^{\tau+1})^j} + \frac{1}{L}\ln(1-p^\tau) + \frac{p^\tau}{L} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L}$$

$$= \frac{1}{L}\sum_{i\geq 2}\frac{p^i\mathbf{R}_i}{i} - \frac{1}{L}\sum_{j\geq 2}\frac{q^j\mathbf{R}_j}{j}\frac{p^{j\tau}}{(1-p^{\tau+1})^j} + \frac{1}{L}\ln(1-p^\tau) + \frac{p^\tau}{L} + \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L}.$$

This is the expansion that was also obtained in [7].

D Total number of null rounds

Notations:

$$\begin{aligned}\mathbf{P}_T(i,t) &:= \text{Probability that, starting with } i \text{ players, we end with } t \text{ null rounds,}\\ \tilde{\mathbf{P}}_T(i,t) &:= \text{Probability that, starting with } i \text{ players, we end with } t \text{ null rounds,}\\ &\quad\ \text{given that the } i \text{ players were obtained in a null round.}\end{aligned}$$

We have the recurrences:

$$\mathbf{P}_T(1,0) = 1, \quad \tilde{\mathbf{P}}_T(1,1) = 1,$$

$$\mathbf{P}_T(i,t) = (p^i)^\tau [\![t=\tau]\!] + \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{P}_T(\ell,t-s),$$

$$\tilde{\mathbf{P}}_T(i,t) = (p^i)^{\tau-1}[\![t=\tau]\!] + \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{P}_T(\ell,t-1-s).$$

$$\mathbf{I}_i = \sum_t \mathbf{P}_T(i,t)t = (p^i)^\tau\tau + \sum_t\sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{P}_T(\ell,t-s)[t-s+s]$$

$$= (p^i)^\tau\tau + \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{I}_\ell + \sum_t\sum_{s=0}^{\tau-1}(p^i)^s s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{P}_T(\ell,t-s),$$

$$= (p^i)^\tau\tau + \Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{I}_\ell + \Sigma_4(i,\tau)[1-(p^i)^\tau]/\Sigma_1(i,\tau)$$

$$= \frac{1-p^{i\tau}}{1-p^i}\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{I}_\ell + \frac{p^i\left(1-p^{i\tau}\right)}{1-p^i}, \quad \mathbf{I}_1 = 0.$$

$$\begin{aligned}
\tilde{\mathbf{I}}_i &= \sum_t \tilde{\mathbf{P}}_T(i,t)t = (p^i)^{\tau-1}\tau + \sum_t \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} \mathbf{P}_T(\ell, t-1-s)[t-1-s+s+1] \\
&= (p^i)^{\tau-1}\tau + \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} \mathbf{I}_\ell + \sum_t \sum_{s=0}^{\tau-2} (p^i)^s (s+1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} \mathbf{P}_T(\ell, t-1-s) \\
&= (p^i)^{\tau-1}\tau + \Sigma_1(i,\tau-1)\left[\mathbf{I}_i - (p^i)^\tau \tau - \Sigma_4(i,\tau)[1-(p^i)^\tau]/\Sigma_1(i,\tau)\right]/\Sigma_1(i,\tau) \\
&\qquad\qquad\qquad\qquad + \Sigma_2(i,\tau-1)[1-(p^i)^\tau]/\Sigma_1(i,\tau) \\
&= \frac{1-p^i+p^{i(\tau+1)}-p^{i\tau}}{(1-p^{i\tau})(1-p^i)} + \frac{1-p^i-p^{i(\tau-1)}+p^{i\tau}}{(1-p^{i\tau})(1-p^i)}\mathbf{I}_i \\
&= 1 + \frac{1-p^{i(\tau-1)}}{1-p^{i\tau}}\mathbf{I}_i, \quad \tilde{\mathbf{I}}_1 = 1.
\end{aligned}$$

The mean is given by

$$\mathbf{I}_n \sim \sum_{i=2}^{\infty} \frac{p^i}{Li}\tilde{\mathbf{I}}_i + \sum_{l\neq 0}\sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{\mathbf{I}}_i e^{-2l\pi\mathbf{i}\{\log n^*\}}.$$

Indeed, $\mathbf{I}_n$ from the paper [7] satisfies the same recursion as here, after unwinding it as shown in the previous example.

And now we look at the nonfluctuating part in the asymptotic expansion of the mean:

$$\begin{aligned}
\sum_{i=2}^{\infty} \frac{p^i}{Li}\tilde{\mathbf{I}}_i &= \frac{1}{L}\sum_{n\geq 2} \frac{p^n}{n}\Big(1 + \frac{1-p^{n(\tau-1)}}{1-p^{n\tau}}\mathbf{I}_n\Big) \\
&= \frac{1}{L}(-\ln(1-p)-p) + \frac{1}{L}\sum_{n\geq 2}\frac{p^n}{n}\frac{1-p^{n\tau}+p^{n\tau}-p^{n(\tau-1)}}{1-p^{n\tau}}\mathbf{I}_n \\
&= 1 - \frac{p}{L} + \frac{1}{L}\sum_{n\geq 2}\frac{p^n}{n}\mathbf{I}_n - \frac{1}{L}\sum_{n\geq 2}\frac{p^{n\tau}}{n}\frac{1-p^n}{1-p^{n\tau}}\mathbf{I}_n \\
&= 1 - \frac{p}{L} + \frac{1}{L}\sum_{n\geq 1}\frac{p^n}{n}\mathbf{I}_n - \frac{1}{L}\sum_{n\geq 2}\frac{p^{n\tau}}{n}\Bigg(\sum_{1\leq j\leq n}\binom{n}{j}q^j p^{n-j}\mathbf{I}_j + p^n\Bigg) \\
&= 1 - \frac{p}{L} + \frac{1}{L}\sum_{n\geq 1}\frac{p^n}{n}\mathbf{I}_n - \frac{1}{L}\sum_{n\geq 1}\frac{p^{n(\tau+1)}}{n} + \frac{p^{\tau+1}}{L} - \frac{1}{L}\sum_{n\geq 1}\frac{p^{n\tau}}{n}\sum_{1\leq j\leq n}\binom{n}{j}q^j p^{n-j}\mathbf{I}_j \\
&= 1 - \frac{p}{L} + \frac{1}{L}\ln(1-p^{\tau+1}) + \frac{p^{\tau+1}}{L} + \frac{1}{L}\sum_{n\geq 1}\frac{p^n}{n}\mathbf{I}_n - \frac{1}{L}\sum_{j\geq 1}\frac{q^j\mathbf{I}_j}{j}\sum_{n\geq j}\binom{n-1}{j-1}p^{n\tau}p^{n-j} \\
&= 1 - \frac{p}{L} + \frac{1}{L}\ln(1-p^{\tau+1}) + \frac{p^{\tau+1}}{L} + \frac{1}{L}\sum_{n\geq 1}\frac{p^n}{n}\mathbf{I}_n - \frac{1}{L}\sum_{j\geq 1}\frac{q^j\mathbf{I}_j}{j}\frac{p^{j\tau}}{(1-\tau^{\tau+1})^j}.
\end{aligned}$$

This is the expression given in [7].

E Total number of leftovers

We have here only the failure case. This gives

$$\begin{aligned}
L'_i &= \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^ip^{i\tau}}i + \Sigma_1(i,\tau)\left[\sum_{\ell=0}^{i-1}\binom{i}{\ell}q^\ell p^{i-\ell}L'_\ell + q^i\left[L'_i - \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^ip^{i\tau}}i\right]\right] \\
\tilde{L}'_i &= \left[p^{i(\tau-1)} + \frac{1-p^{i(\tau-1)}}{1-p^i}q^i\frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^ip^{i\tau}}\right]i + \frac{1-p^{i(\tau-1)}}{1-p^{i\tau}}\left[L'_i - \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^ip^{i\tau}}i\right]
\end{aligned}$$

$$= \frac{1-p^{i(\tau-1)}}{1-p^{i\tau}} L'_i + \frac{p^{i(\tau-1)}(1-p^i)}{1-p^{i\tau}} i.$$

The mean is given by

$$L'_n \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{L}'_i + \sum_{l\neq 0} \sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!} \tilde{L}'_i e^{-2l\pi \mathbf{i}\{\log n^*\}}.$$

The recursion derived in [7] is

$$L'_n = \frac{1-p^{n\tau}}{1-p^n} \sum_{j=1}^{n} \binom{n}{j} p^{n-j} q^j L'_j + np^{n\tau}.$$

Here, we have

$$\begin{aligned}
L'_n &= \frac{p^{n\tau}(1-p^n)}{1-p^n-q^n+q^np^{n\tau}} n + \frac{1-p^{n\tau}}{1-p^n} \left[\sum_{j=1}^{n-1} \binom{n}{j} q^j p^{n-j} L'_j + q^n \left(L'_n - \frac{p^{n\tau}(1-p^n)}{1-p^n-q^n+q^np^{n\tau}} n \right) \right] \\
&= \frac{p^{n\tau}(1-p^n)}{1-p^n-q^n+q^np^{n\tau}} n + \frac{1-p^{n\tau}}{1-p^n} \left[\sum_{j=1}^{n} \binom{n}{j} q^j p^{n-j} L'_j - q^n \frac{p^{n\tau}(1-p^n)}{1-p^n-q^n+q^np^{n\tau}} n \right] \\
&= \frac{p^{n\tau}(1-p^n)}{1-p^n-q^n+q^np^{n\tau}} n \left[1 - \frac{1-p^{n\tau}}{1-p^n} q^n \right] + \frac{1-p^{n\tau}}{1-p^n} \sum_{j=1}^{n} \binom{n}{j} q^j p^{n-j} L'_j \\
&= p^{n\tau} n + \frac{1-p^{n\tau}}{1-p^n} \sum_{j=1}^{n} \binom{n}{j} q^j p^{n-j} L'_j,
\end{aligned}$$

and hence we do have the same recursion.

Now we turn to the nonfluctuating part of the mean:

$$\begin{aligned}
\sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{L}'_i &= \sum_{i\geq 2} \frac{p^i}{Li} \left[\frac{1-p^{i(\tau-1)}}{1-p^{i\tau}} L'_i + \frac{p^{i(\tau-1)}(1-p^i)}{1-p^{i\tau}} i \right] \\
&= \frac{1}{L} \sum_{i\geq 2} \frac{p^i}{i} \frac{1-p^{i\tau}+p^{i\tau}-p^{i(\tau-1)}}{1-p^{i\tau}} L'_i + \frac{1}{L} \sum_{i\geq 2} \frac{p^{i\tau}(1-p^i)}{1-p^{i\tau}} \\
&= \frac{1}{L} \sum_{i\geq 2} \frac{p^i}{i} L'_i - \frac{1}{L} \sum_{i\geq 2} \frac{p^{i\tau}}{i} \left[\sum_{j=1}^{i} \binom{i}{j} p^{i-j} q^j L'_j + \frac{1-p^i}{1-p^{i\tau}} i p^{i\tau} \right] + \frac{1}{L} \sum_{i\geq 2} \frac{p^{i\tau}(1-p^i)}{1-p^{i\tau}} \\
&= \frac{1}{L} \sum_{i\geq 2} \frac{p^i}{i} L'_i - \frac{1}{L} \sum_{i\geq 2} \frac{p^{i\tau}}{i} \sum_{j=1}^{i} \binom{i}{j} p^{i-j} q^j L'_j - \frac{1}{L} \sum_{i\geq 2} \frac{1-p^i}{1-p^{i\tau}} p^{2i\tau} + \frac{1}{L} \sum_{i\geq 2} \frac{p^{i\tau}(1-p^i)}{1-p^{i\tau}} \\
&= \frac{1}{L} \sum_{i\geq 2} \frac{p^i}{i} L'_i - \frac{1}{L} \sum_{j\geq 1} \frac{q^j L'_j}{j} \sum_{i\geq j} p^{i\tau} \binom{i-1}{j-1} p^{i-j} + \frac{1}{L} \sum_{i\geq 2} p^{i\tau}(1-p^i) \\
&= \frac{1}{L} \sum_{i\geq 2} \frac{p^i}{i} L'_i - \frac{1}{L} \sum_{j\geq 1} \frac{q^j L'_j}{j} \frac{q^{j\tau}}{(1-p^{\tau+1})^j} + \frac{1}{L} \frac{1}{1-p^\tau} - \frac{1}{L} \frac{1}{1-p^{\tau+1}} - qp^\tau,
\end{aligned}$$

which is the same expression as in [7].

F Matrix expressions.

We will give a few explicit matrix expressions for several quantities computed before. We will not use these expressions, but we only wanted to show that some compact relations can be written down in some cases, showing an unified view of our different RV.

F.1 Success probability

Let

$$\Pi[i,u] := \Sigma_1(i,\tau)\binom{i}{u} q^u p^{i-u}, \ \ i,u \geq 2,$$

and

$$\varphi_1(i) := \Sigma_1(i,\tau)\binom{i}{1} q^1 p^{i-1}, \ \ i \geq 2.$$

Then we have the expression

$$P(.) = \sum_{k=0}^{\infty} \Pi^k \varphi_1 = [I - \Pi]^{-1}\varphi_1.$$

Note that, to get some precision in S_n, only finite matrices are necessary.

F.2 Number of rounds

Let

$$\varphi_2(i) := \Sigma_2(i,\tau)P(i)/\Sigma_1(i,\tau), \ \ i \geq 2.$$

Then we have the expression

$$R_. = \Pi R_. + \varphi_2 = [I - \Pi]^{-1}\varphi_2.$$

F.3 Number of null rounds

Let

$$\varphi_3(i) := \Sigma_4(i,\tau)P(i)/\Sigma_1(i,\tau), \ \ i \geq 2.$$

Then we have the expression

$$I_. = [I - \Pi]^{-1}\varphi_3.$$

F.4 Number of leftovers

Fix r. Let

$$\Pi_1[i,u] := \Sigma_1(i,\tau)\binom{i}{u} q^u p^{i-u}, \ \ i,u > r,$$

and

$$\varphi_3(i) := \Sigma_1(i,\tau)\binom{i}{r} q^r p^{i-r} P'_W(r,r).$$

Then we have the expression

$$P'_W(.,r) = [I - \Pi_1]^{-1}\varphi_3.$$

G Model 2

We will only briefly mention the modifications related to the main expressions. A supplementary last index will indicate how many null rounds are allowed before failure. Only the mean in the success case will be given, all other cases can be similarly computed; we leave the details for any research student who is interested.

$$P(i,\tau) = \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P(\ell, \tau - s), \ \ i \geq 2,$$

$$\tilde{P}(i,\tau) = \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P(\ell, \tau - s - 1) = P(i, \tau - 1),$$

$$
\begin{aligned}
P_K(i,k,\tau) &= \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P_K(\ell, k-1-s, \tau-s),\ i \geq 2,\\
\tilde{P}_K(i,k,\tau) &= \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P_K(\ell, k-1-s, \tau-s-1) = P_K(i,k,\tau-1),\\
R_i(\tau) &= \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} R_\ell(\tau-s) + \sum_{s=0}^{\tau-1}(p^i)^s (s+1) \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P(\ell, \tau-s),\\
\tilde{R}_i(\tau) &= R_i(\tau-1),\\
N_i(\tau) &= \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} [si + i + N_\ell(\tau-s)],\\
\tilde{N}_i(\tau) &= N_i(\tau-1),\\
P_T(i,t,\tau) &= \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-s, \tau-s),\ i \geq 2,\\
\tilde{P}_T(i,t) &= \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-1-s, \tau-s-1),\ i \geq 2,\\
I_i(\tau) &= \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} I_\ell(\tau-s) + \sum_{s=0}^{\tau-1}(p^i)^s s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P(\ell, \tau-s)\\
\tilde{I}_i &= \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} I_\ell(\tau-s-1) + \sum_{s=0}^{\tau-2}(p^i)^s (s+1) \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P(\ell, \tau-s-1),\\
P'_W(r,r,\tau) &= (p^r)^\tau + \Sigma_1(r,\tau) q^r P'_W(r,r,\tau),\\
\tilde{P}'_W(r,r,\tau) &= P'_W(r,r,\tau-1),\\
P'_W(i,r,\tau) &= \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=r}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P'_W(\ell, r, \tau-s), i > r,\ i \geq 2,\\
\tilde{P}'_W(i,r,\tau) &= \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=r}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P'_W(\ell, r, \tau-s-1) = P'_W(i,r,\tau-1),\\
L'_i(\tau) &= \sum_{s=0}^{\tau-1}\sum_{\ell=r}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} \sum_{r=0}^{i-1} P'_W(\ell, r, \tau-s) r + [(p^i)^\tau + q^i P'_W(i,i,\tau)] i,\\
\tilde{L}'_i &= L'_i(\tau-1).
\end{aligned}
$$

One can now proceed as in Model 1.

H Proof of Theorem 8.2

We start from

$$
C_{n,S} = [\![I=1]\!][S_{\Gamma,1}(J,G) + JG] + [\![I>1]\!][S_{\Gamma,2}(J,I) + JI + \tilde{C}_{I,S}].
$$

Here, $S_{\Gamma,1}, \eta_1$ are related to the case $I = 1$ and $S_{\Gamma,2}, \eta_2$ are related to the case $I > 1$. In the sequel, with some abuse of notation, $\mathcal{O}_V(1)$ will denote a RV, asymptotically independent of n, with finite moments. Rewriting,

$$
C_{n,S} = \sum_j \left(\mathbb{P}[J=j, I=1][S_{\Gamma,1}(j,G) + jG] + \sum_{i\geq 2} \mathbb{P}[J=j, I=i] \left[[S_{\Gamma,2}(j,i) + ji] \tilde{P}(i) + \tilde{C}_{i,S} \right] \right).
$$

We have, *conditioned on a success*, (we use the dominant success probability $Pd(S)$)

$$\begin{aligned}
\frac{C_{n,S}}{Pd(S)} &\sim \frac{\Pi_1}{Pd(S)} \sum_j \frac{\mathbb{P}[J=j, I=1]}{\Pi_1}[S_{\Gamma,1}(j,G)+jG] \\
&+ \frac{V_1}{Pd(S)} \sum_j \sum_{i\geq 2} \frac{\mathbb{P}[J=j, I=i]\tilde{P}(i)}{V_1}[S_{\Gamma,2}(j,i)+ji] \\
&+ \frac{V_1}{Pd(S)} \sum_j \sum_{i\geq 2} \frac{\mathbb{P}[J=j, I=i]}{V_1}\tilde{C}_{i,S} \\
&= \frac{\Pi_1}{Pd(S)} \sum_j \frac{\mathbb{P}[J=j, I=1]}{\Pi_1}[S_{\Gamma,1}(j,G)+jG] \\
&+ \frac{V_1}{Pd(S)} \sum_j \sum_{i\geq 2} \frac{\mathbb{P}[J=j, I=i]\tilde{P}(i)}{V_1}[S_{\Gamma,2}(j,i)+ji] + \mathcal{O}_V(1).
\end{aligned}$$

Again we will separate the contribution of urn J from that of urns $< J$. So, conditioning on $J = j$ and $\Gamma_k(j)$ denoting a sequence of iid truncated geometric RV $< j$,

$$\begin{aligned}
S_{\Gamma,1}(j,G) + jG &= S_{\Gamma,1}(j,0) - \sum_{j=1}^{G} \Gamma_k(j) + jG \\
&= S_{\Gamma,1}(j,0) + \mathcal{O}_V(1) + (\log n^* + \eta_1)G \\
&= S_{\Gamma,1}(j,0) + \mathcal{O}_V(1) + \log n^* \cdot \mathcal{O}_V(1),
\end{aligned}$$

and similarly for $S_{\Gamma,2}(j,i) + ji$. So

$$\begin{aligned}
\frac{C_{n,S}}{Pd(S)} &\sim \frac{\Pi_1}{Pd(S)} \sum_j \frac{\mathbb{P}[J=j, I=1]}{\Pi_1} S_{\Gamma,1}(j,0) \\
&+ \frac{V_1}{Pd(S)} \sum_j \sum_{i\geq 2} \frac{\mathbb{P}[J=j, I=i]\tilde{P}(i)}{V_1} S_{\Gamma,2}(j,0) + \mathcal{O}_V(1) + \log n^* \cdot \mathcal{O}_V(1).
\end{aligned}$$

Now we must show that $S_\Gamma(j,0)$ is asymptotically Gaussian. We could simply use Feller [2, example *IX,1,a* on triangular arrays], but we want an error estimation. We will provide the first terms of our expansions, but Maple "knows" more. The standard deviation of $\Gamma(j)$ will be denoted by $\sigma(j)$. We have

$$\begin{aligned}
\Sigma_0 &= 1 - \frac{e^{-L\eta}}{np}, \\
E(j) &\sim \frac{1}{p} - \frac{e^{-L\eta}(j-1)}{np}, \\
\sigma(j) &\sim \frac{\sqrt{q}}{p} - \frac{e^{-L\eta}(j-1)^2}{2n\sqrt{q}}.
\end{aligned}$$

Now the probability generating function (PGF) of $\Gamma(j)$ is given by

$$F(z) = \frac{1}{\Sigma_0} \sum_{l=1}^{j-1} pq^{l-1} z^l = \frac{1}{\Sigma_0} \left[\frac{pz}{1-qz} - \frac{e^{-L\eta} z^j}{n(1-qz)} \right],$$

and the PGF of $S_\Gamma(j,0)$ is given by $[F(z)]^n$. We will now use the Saddle point method (for a good introduction to this method, see Flajolet and Sedgewick [3, chapter VIII]). By Cauchy's theorem,

$$\mathbb{P}(S_\Gamma(j,0) = k) = \frac{1}{2\pi \mathbf{i}} \int_\Omega \frac{[F(z)]^n}{z^{k+1}} dz = \frac{1}{2\pi \mathbf{i}} \int_\Omega e^{H(z)} dz,$$

where Ω is inside the domain of analyticity of the integrand and encircles the origin and

$$H(z) = n\left(\ln\left[\frac{pz}{1-qz} - \frac{e^{-L\eta}z^j}{n(1-qz)}\right] - \ln(\Sigma_0)\right) - (k+1)\ln(z).$$

Set

$$H^{(i)} := \frac{d^i H}{dz^i}.$$

First we must find the solution of

$$H^{(1)}(\tilde{z}) = 0 \tag{H.1}$$

with smallest modulus.

Set $\tilde{z} := z^* - \varepsilon$, where $z^* = \lim_{n\to\infty} \tilde{z}$. Here, it is easy to check that $z^* = 1$. Set $k = nE(j) + \sqrt{n}\sigma(j)x$, x fixed. We will soon see that $\varepsilon = \mathcal{O}\big(\frac{1}{\sqrt{n}}\big)$, so we can expand z^j in $F(z)$ as

$$z^j = 1 - j\varepsilon + \frac{j(j-1)}{2}\varepsilon^2 + \cdots$$

Also $j = \log n^* + \eta$. This leads, to first order (keeping only the ε term in (H.1)), to

$$\varepsilon := \frac{-px}{\sqrt{nq}} + \mathcal{O}\Big(\frac{\log n^*}{n}\Big).$$

This shows that, asymptotically, ε is given by a series of powers of $n^{-1/2}$, where each coefficient is given by a series of powers of $\log n^*$. To obtain more precision, we set again $k = nE(j) + \sqrt{n}\sigma(j)x$, expand in powers of $n^{-1/2}$, and equate each coefficient to 0. We have, with $\tilde{z} := z^* - \varepsilon = 1 - \varepsilon$,

$$\mathbb{P}(S_\Gamma(j,0) = k) = \frac{1}{2\pi\mathbf{i}}\int_\Omega \exp\Big[H(\tilde{z}) + H^{(2)}(\tilde{z})(z-\tilde{z})^2/2! + \sum_{l=3}^{\infty} H^{(l)}(\tilde{z})(z-\tilde{z})^l/l!\Big]dz.$$

Note that the linear term vanishes. Set $z = \tilde{z} + \mathbf{i}\tau$. This gives

$$\mathbb{P}(S_\Gamma(j,0) = k) \sim \frac{1}{2\pi}\exp[H(\tilde{z})]\int_{-\infty}^{\infty} \exp\Big[H^{(2)}(\tilde{z})(\mathbf{i}\tau)^2/2! + \sum_{l=3}^{\infty} H^{(l)}(\tilde{z})(\mathbf{i}\tau)^l/l!\Big]d\tau. \tag{H.2}$$

Let us first analyze $H(\tilde{z})$. We obtain

$$H(\tilde{z}) = -x^2/2 + \mathcal{O}\Big(\frac{1}{\sqrt{n}}\Big).$$

Also,

$$H^{(2)}(\tilde{z}) = n\frac{q}{p^2} + \mathcal{O}(\sqrt{n}\,),$$
$$H^{(4)}(\tilde{z}) = \mathcal{O}(n).$$

We can now compute (H.2), for instance by using the classical trick of setting

$$S^{(2)}(\tilde{z})(i\tau)^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(i\tau)^l/l! = -u^2/2.$$

Computing τ as a truncated series in u, this gives, by inversion,

$$\tau = \frac{u}{\sqrt{nq/p^2}} + u^2\mathcal{O}\Big(\frac{1}{n}\Big).$$

Setting $d\tau = \frac{d\tau}{du}du$, and integrating on $-\infty < u < \infty$, this gives

$$\frac{1}{\sqrt{2\pi nq/p^2}}\left[1+\mathcal{O}\Big(\frac{1}{\sqrt{n}}\Big)\right].$$

Finally (H.2) leads to

$$\mathbb{P}(S_\Gamma(j,0)=k) \sim \frac{1}{\sqrt{2\pi nq/p^2}}e^{-x^2/2}\left[1+\mathcal{O}\Big(\frac{1}{\sqrt{n}}\Big)\right].$$

Now we consider

$$\begin{aligned}
&\mathbb{P}\left(\frac{\frac{C_{n,S}}{Pd(S)}-\frac{\mathbb{E}(C_{nS})}{Pd(S)}}{\sqrt{nq/p^2}}\le x\right)\\
&\sim \mathbb{P}\Bigg(\frac{\frac{\Pi_1}{Pd(S)}\sum_j \frac{\mathbb{P}[J=j,I=1]}{\Pi_1}[S_{\Gamma,1}(j,0)-nE_1(j)] + \frac{V_1}{Pd(S)}\sum_j\sum_{i\ge 2}\frac{\mathbb{P}[J=j,I=i]\tilde{P}(i)}{V_1}[S_{\Gamma,2}(j,0)-nE_2(j)]}{\sqrt{nq/p^2}}\\
&\qquad\qquad + \frac{\mathcal{O}_V(1)+\log n^*\cdot\mathcal{O}_V(1)}{\sqrt{nq/p^2}}\le x\Bigg)\\
&\sim \mathbb{P}\Bigg(\frac{\frac{\Pi_1}{Pd(S)}\sum_j \frac{\mathbb{P}[J=j,I=1]}{\Pi_1}[S_{\Gamma,1}(j,0)-nE_1(j)]}{\sqrt{n}\sigma_1(j)}\\
&\qquad\qquad + \frac{\frac{V_1}{Pd(S)}\sum_j\sum_{i\ge 2}\frac{\mathbb{P}[J=j,I=i]\tilde{P}(i)}{V_1}[S_{\Gamma,2}(j,0)-nE_2(j)]}{\sqrt{n}\sigma_2(j)}\le x\Bigg)
\end{aligned}$$

as

$$\frac{\sigma(j)}{\sqrt{q/p^2}}\xrightarrow[n\to\infty]{} 1.$$

Now

$$\frac{\mathbb{V}(C_n)}{Pd(S)nq/p^2}\xrightarrow[n\to\infty]{} 1,$$

which concludes the proof.

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