q–ENGEL SERIES EXPANSIONS AND SLATER'S IDENTITIES

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Abstract. We describe the q–Engel series expansion for Laurent series discovered by John Knopfmacher and use this algorithm to shed new light on partition identities related to two entries from Slater's list. In our study Al-Salam/Ismail and Santos polynomials play a crucial rôle.

Dedicated to the memory of John Knopfmacher 1937–1999

1. Introduction

In 1987 John Knopfmacher conceived of the idea of representing formal Laurent series as sums of reciprocals of polynomials. The initial motivation was some old results on representations for real numbers by sums of reciprocals of integers, due originally to Lambert, Engel and Sylvester. John together with Arnold had previously investigated various extensions of the real number representations (see the article by Kalpazidou and Ganatsiou [10] in this issue). However the development of analogous expansions for Laurent Series turned out to have some unexpected benefits.

Around the time of publication of [11] it was noticed that a number of famous expansions including those of Euler and the Rogers–Ramanujan identities were, in fact, special cases of the q–Engel expansion. This led to the interesting project of using the q–Engel algorithm (described below) to provide new proofs of these identities. This gave rise to a first paper by George, Arnold and John [4] in which the Rogers–Ramanujan identities and some identities of Euler were given new inductive proofs using the q–Engel algorithm discovered by John. Subsequently George, Arnold and Peter have continued these investigations leading to the further publications [3, 6, 5].

To explain our new results we begin by recalling the q–Engel expansion [11, 12] for the field $\mathcal{L} = \mathbb{C}((q))$ of formal Laurent series over the complex numbers, $\mathbb{C}$. If

$$A = \sum_{n=\nu}^{\infty} L_n q^n \quad \text{with} \quad L_\nu \neq 0,$$

we call $\nu = \nu(A)$ the order of A and we define the norm of A to be

$$\|A\| = 2^{-\nu(A)}.$$

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In addition, we define the INTEGRAL PART of A by

$$[A] = \sum_{\nu \le n \le 0} L_n q^n \,. \tag{1}$$

Engel (c. f. [14, §34]) originally defined a series expansion for real numbers. In [11], this concept was extended to $\mathfrak{L}$ in the following way:

Theorem 1. [q–Engel Expansion Theorem ([11, th. 1. 4]).] *Every $A \in \mathfrak{L}$ has a finite or convergent (relative to the above norm) series expansion of the form*

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n} \,, \tag{2}$$

where $a_n \in \mathbb{C}[q^{-1}]$, $a_0 = [A]$,

$$\nu(a_n) \le -n \,, \text{ and } \nu(a_{n+1}) \le \nu(a_n) - 1 \,. \tag{3}$$

The series (2) is unique for A, and it is finite if and only if $A \in \mathbb{C}(q)$. In addition, if

$$a_0 + \sum_{j=1}^{n} \frac{1}{a_1 \cdots a_j} = \frac{p_n}{q_n} \,, \qquad \text{where } q_n = a_1 a_2 \cdots a_n \,,$$

then

$$\left\| A - \frac{p_n}{q_n} \right\| \le \frac{1}{2^{n+1} \|q_n\|}$$

and

$$\nu\Big(A - \frac{p_n}{q_n}\Big) = -\nu(q_{n+1}) \ge \frac{(n+1)(n+2)}{2} \,.$$

In fact, the a_n (the "digits") are for $n \ge 1$ given recursively by

$$a_n = \left[\frac{1}{A_n} \right] \tag{4}$$

where $A_0 = A$, $a_0 = [A]$, $A_1 = A - a_0$, and for $n \ge 1$

$$A_{n+1} = a_n A_n - 1. \tag{5}$$

□

To illustrate for example how the first Rogers–Ramanujan identity represents a q–Engel expansion, we write it as

$$\prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}$$
$$= 1 + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n} \,,$$

where $a_n = (1-q^n)/q^{2n-1}$ for $n \ge 1$.

In Section 2, we will in fact make use of a slight variation of the q–Engel algorithm in which (5) is replaced by

$$A_{n+1} = q(a_n A_n - 1). \tag{6}$$

Corresponding to (2) we have instead the modified expansion,

$$A = a_0 + \sum_{n=1}^{\infty} \frac{q^{-n}}{a_1 a_2 \cdots a_n}\,. \tag{7}$$

An explicit treatment of this modified q–Engel expansion can be found in [6].

In this paper we use the Engel approach e. g. to derive in a new way the representation

$$\lim_{n\to\infty} U_n(0) = (q;q^2)_\infty \sum_{n\geq 0} \frac{q^{2n^2}}{(q;q)_{2n}}$$

with $(x;q)_n := (1-x)(1-xq)\dots(1-xq^{n-1})$ for the limit of specialized Al-Salam/Ismail polynomials defined below.

On the other hand, it has already been shown [7] that the *Santos* polynomials converge as follows

$$S_n \to S_\infty := \prod_{k\geq 1,\ k\equiv \pm 2, \pm 3, \pm 4, \pm 5 \pmod{16}} \frac{1}{1-q^k}.$$

We link the Al-Salam/Ismail polynomials to the Santos polynomials and shed in this way new light on formulæ related to two identities due to Slater one of which being of the form

$$\sum_{n\geq 0} \frac{q^{2n^2}}{(q;q)_{2n}} = \prod_{k\geq 1,\ k\equiv \pm 2, \pm 3, \pm 4, \pm 5 \pmod{16}} \frac{1}{1-q^k}.$$

One of the main results is Theorem 2 which embeds a new generating function relation for Al-Salam/Ismail polynomials into a q–Engel context.

Finally, in Section 3, we discuss Al-Salam/Ismail and Santos polynomials in the context of identities of Garrett/Ismail/Stanton type.

2. Al-Salam/Ismail polynomials and Slater's identities (38) and (39)

The Al-Salam and Ismail polynomials $U_n(x;a,b\,|\,q)$ are defined by [1]

$$U_{-1}(x;a,b\,|\,q) = 0, \qquad U_0(x;a,b\,|\,q) = 1,$$
$$U_n(x;a,b\,|\,q) = x(1+aq^{n-1})U_{n-1}(x;a,b\,|\,q) - bq^{n-2}U_{n-2}(x;a,b\,|\,q), \quad n\geq 1.$$

Al-Salam and Ismail gave the explicit representation

$$\begin{aligned} U_n(x;a,b\,|\,q) &= \sum_{k=0}^{\lfloor n/2\rfloor} \frac{(-a;q)_{n-k}(q;q)_{n-k}x^{n-2k}}{(-a;q)_k(q;q)_k(q;q)_{n-2k}}(-b)^k q^{k(k-1)} \\ &= \sum_{k=0}^{\lfloor n/2\rfloor} (-aq^k;q)_{n-2k}x^{n-2k} \begin{bmatrix} n-k \\ k \end{bmatrix}_q (-b)^k q^{k(k-1)}; \end{aligned}$$

here we used the *Gaussian* polynomials

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

We specialize and consider $U_n(1; -q^{2\alpha+1}, -q^{2\alpha+2} \mid q^2)$, but simply write $U_n(\alpha)$ for that. Let us also rewrite the recursion

$$\begin{aligned} U_{-1}(\alpha) &= 0, \quad U_0(\alpha) = 1, \\ U_n(\alpha) &= (1 - q^{2n+2\alpha-1}) U_{n-1}(\alpha) + q^{2n+2\alpha-2} U_{n-2}(\alpha), \quad n \ge 1 \end{aligned}$$

and the explicit formula

$$U_n(\alpha) = \sum_{k=0}^{\lfloor n/2 \rfloor} \begin{bmatrix} n-k \\ k \end{bmatrix}_{q^2} q^{2k^2+2k\alpha} (q^{2k+1+2\alpha}; q^2)_{n-2k}.$$

In our treatment, α will be either 0 or 1, and we try to treat both cases simultaneously whenever possible.

The recursion can be rewritten as

$$\frac{U_n(\alpha)}{(q^{1+2\alpha}; q^2)_n} = \frac{U_{n-1}(\alpha)}{(q^{1+2\alpha}; q^2)_{n-1}} + \frac{q^{2n-2+2\alpha}}{(q^{1+2\alpha}; q^2)_n} U_{n-2}(\alpha)$$

and summed:

$$\frac{U_n(\alpha)}{(q^{1+2\alpha}; q^2)_n} = 1 + \sum_{k=1}^{n} \frac{q^{2k-2+2\alpha}}{(q^{1+2\alpha}; q^2)_k} U_{k-2}(\alpha).$$

In the limit $n \to \infty$,

$$\frac{U_\infty(\alpha)}{(q^{1+2\alpha}; q^2)_\infty} = 1 + \sum_{k \ge 1} \frac{q^{2k-2+2\alpha}}{(q^{1+2\alpha}; q^2)_k} U_{k-2}(\alpha).$$

Theorem 2. *If one applies the (modified) Engel algorithm to $U_\infty(\alpha)/(q^{1+2\alpha}; q^2)_\infty$ the quantities $A_n(\alpha)$ are given by*

$$A_n(\alpha) = \sum_{k \ge 2} \frac{q^{2kn+2\alpha-1} U_{k-2}(\alpha)}{(q^{2n-1+2\alpha}; q^2)_k} = \sum_{j \ge 0} \frac{q^{2j^2+2\alpha j+4nj+4n+2\alpha-1}}{(q^{2n-1+2\alpha}; q^2)_{j+1} (q^{2n}; q^2)_{j+1}};$$

the digits $a_n(\alpha)$ are given by

$$a_n(\alpha) = \frac{(1 - q^{2n-1+2\alpha})(1 - q^{2n})}{q^{4n-1+2\alpha}} \qquad \textit{for } n \ge 1$$

and $a_0(\alpha) = 1$.

Proof. First, let us prove that the two expressions given for $A_n(\alpha)$ are indeed equal:

$$\begin{aligned}
&\sum_{k\geq 2}\frac{q^{2kn+2\alpha-1}U_{k-2}(\alpha)}{(q^{2n-1+2\alpha};q^2)_k}\\
&=\sum_{k\geq 0}\frac{q^{2(k+2)n+2\alpha-1}}{(q^{2n-1+2\alpha};q^2)_{k+2}}\sum_{0\leq 2j\leq k}\begin{bmatrix}k-j\\ j\end{bmatrix}_{q^2}q^{2j^2+2j\alpha}(q^{2j+1+2\alpha};q^2)_{k-2j}\\
&=\sum_{j,k\geq 0}\frac{q^{2(k+2j+2)n+2\alpha-1}}{(q^{2n-1+2\alpha};q^2)_{k+2j+2}}\begin{bmatrix}k+j\\ j\end{bmatrix}_{q^2}q^{2j^2+2j\alpha}(q^{2j+1+2\alpha};q^2)_k\\
&=\sum_{j,k\geq 0}\frac{q^{2(2j+2)n+2\alpha-1+2nk+2j^2+2j\alpha}}{(q^{2n-1+2\alpha};q^2)_{2j+2}(q^{2n+4j+3+2\alpha};q^2)_k}\frac{(q^{2j+2};q^2)_k}{(q^2;q^2)_k}(q^{2j+1+2\alpha};q^2)_k\\
&=\sum_{j\geq 0}\frac{q^{2(2j+2)n+2\alpha-1+2j^2+2j\alpha}}{(q^{2n-1+2\alpha};q^2)_{2j+2}}\sum_{k\geq 0}\frac{(q^{2j+2};q^2)_k(q^{2j+1+2\alpha};q^2)_k q^{2nk}}{(q^2;q^2)_k(q^{2n+4j+3+2\alpha};q^2)_k}
\end{aligned}$$

(the inner sum can be computed by q–Gauss [2, p.20])

$$\begin{aligned}
&=\sum_{j\geq 0}\frac{q^{2(2j+2)n+2\alpha-1+2j^2+2j\alpha}}{(q^{2n-1+2\alpha};q^2)_{2j+2}}\frac{(q^{2n+2\alpha+2j+1};q^2)_\infty(q^{2n+2j+2};q^2)_\infty}{(q^{2n+2\alpha+4j+3};q^2)_\infty(q^{2n};q^2)_\infty}\\
&=\sum_{j\geq 0}\frac{q^{4jn+4n+2\alpha-1+2j^2+2j\alpha}}{(q^{2n-1+2\alpha};q^2)_{2j+2}}\frac{(q^{2n+2\alpha+2j+1};q^2)_{j+1}}{(q^{2n};q^2)_{j+1}}\\
&=\sum_{j\geq 0}\frac{q^{4jn+4n+2\alpha-1+2j^2+2j\alpha}}{(q^{2n-1+2\alpha};q^2)_{j+1}(q^{2n};q^2)_{j+1}},
\end{aligned}$$

as desired.

Let us now prove the announced formula for the digits.

This is particularly easy since the $j=0$ term *is* the reciprocal of $a_n(\alpha)$, and therefore

$$\sum_{j\geq 0}\frac{q^{2j^2+2\alpha j+4nj+4n+2\alpha-1}}{(q^{2n-1+2\alpha};q^2)_{j+1}(q^{2n};q^2)_{j+1}}=\frac{1}{a_n}+O(q^{8n+1+4\alpha})=\frac{1}{a_n}\Big(1+O(q^{4n+2\alpha+2})\Big).$$

The recursion is also simple, since

$$\begin{aligned}
&q\big(a_n(\alpha)A_n(\alpha)-1\big)\\
&=q\left(a_n(\alpha)\sum_{j\geq 0}\frac{q^{2j^2+2\alpha j+4nj+4n+2\alpha-1}}{(q^{2n-1+2\alpha};q^2)_{j+1}(q^{2n};q^2)_{j+1}}-1\right)\\
&=qa_n(\alpha)\sum_{j\geq 1}\frac{q^{2j^2+2\alpha j+4nj+4n+2\alpha-1}}{(q^{2n-1+2\alpha};q^2)_{j+1}(q^{2n};q^2)_{j+1}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\ge 1} \frac{q^{2j^2+2\alpha j+4nj+1}}{(q^{2n+1+2\alpha};q^2)_j(q^{2n+2};q^2)_j}\\
&= \sum_{j\ge 0} \frac{q^{2(j+1)^2+2\alpha(j+1)+4n(j+1)+1}}{(q^{2n+1+2\alpha};q^2)_{j+1}(q^{2n+2};q^2)_{j+1}}\\
&= \sum_{j\ge 0} \frac{q^{2j^2+2\alpha j+4(n+1)j+4(n+1)+2\alpha-1}}{(q^{2n+1+2\alpha};q^2)_{j+1}(q^{2n+2};q^2)_{j+1}}\\
&= A_{n+1}(\alpha),
\end{aligned}
$$

as desired. □

Thus our Engel proof established the following relations.

Theorem 3. *For $\alpha = 0, 1$ we have*

$$\lim_{n\to\infty} U_n(\alpha) = (q;q^2)_\infty \sum_{n\ge 0} \frac{q^{2n^2+2n\alpha}}{(q;q)_{2n+\alpha}}.$$

This is a new proof of known results due to Al-Salam and Ismail [1].

Slater's identities (39) and (38)[1] [16] are the formulæ

$$
\begin{aligned}
\sum_{n\ge 0} \frac{q^{2n^2}}{(q;q)_{2n}} &= \prod_{k\ge 1,\ k\equiv \pm 2,\pm 3,\pm 4,\pm 5 \pmod{16}} \frac{1}{1-q^k},\\
\sum_{n\ge 0} \frac{q^{2n^2+2n}}{(q;q)_{2n+1}} &= \prod_{k\ge 1,\ k\equiv \pm 1,\pm 4,\pm 6,\pm 7 \pmod{16}} \frac{1}{1-q^k}.
\end{aligned}
$$

Let us consider the Santos polynomials in the representation

$$
\begin{aligned}
S_n &= \sum_{0\le 2j\le n} q^{2j^2} \begin{bmatrix} n \\ 2j \end{bmatrix}_q,\\
T_n &= \sum_{0\le 2j\le n-1} q^{2j^2+2j} \begin{bmatrix} n \\ 2j+1 \end{bmatrix}_q.
\end{aligned}
$$

Originally these polynomials were studied in [15] and were discussed further in [7]. As pointed out ibid. they possess the following limit property,

$$
\begin{aligned}
S_n \to S_\infty &:= \prod_{k\ge 1,\ k\equiv \pm 2,\pm 3,\pm 4,\pm 5 \pmod{16}} \frac{1}{1-q^k},\\
T_n \to T_\infty &:= \prod_{k\ge 1,\ k\equiv \pm 1,\pm 4,\pm 6,\pm 7 \pmod{16}} \frac{1}{1-q^k}.
\end{aligned}
$$

In [6], the quantities S_∞ and T_∞ were undergone an Engel treatment; see also Section 3 below.

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[1] Observe the order!

After observing these limits, Slater's identities (39) and (38) are immediate by taking $n \to \infty$ in the polynomial representations of S_n and T_n above. However, the Engel approach encoded by Theorem 2 admits another link to Slater's identities which is briefly explained as follows.

Although it is not shown explicitly in [7], it is easy to verify that the polynomials S_n and T_n satisfy the following defining recurrences

$$\begin{aligned} S_n &= S_{n-1} + q^n T_{n-1}, \qquad & S_0 &= 1, \\ T_n &= T_{n-1} + q^{n-1} S_{n-1}, \qquad & T_0 &= 0. \end{aligned}$$

As a direct consequence of these recurrences we see that

$$S_\infty = 1 + \sum_{j\geq 1} q^j T_{j-1} \quad \text{ and } \quad (1-q)\,T_\infty = 1 + \sum_{j\geq 1} q^{2j} T_{j-1}.$$

So, by Theorem 2 we obtain an alternative statement being equivalent to Slater's identities; namely,

$$\frac{1}{q} A_1(\alpha) = \sum_{j\geq 1} q^{(1+\alpha)j} T_{j-1} \qquad (\alpha = 0, 1)$$

where $A_1(\alpha)$ is chosen as in Theorem 2. As we shall see below, this fact is easily established, and—again as a by-product of the Engel context of Theorem 2—one obtains the more general relation

$$A_n(\alpha) = \sum_{j\geq 1} q^{(2n-1+\alpha)j+1} T_{j-1} \qquad (\alpha = 0, 1)$$

where $A_n(\alpha)$ is chosen as in Theorem 2. However, the proof for $n = 1$ is at the same level of complexity as for n, and so we do the computation for general n.

$$\begin{aligned} \sum_{j\geq 1} q^{(2n-1+\alpha)j+1} T_{j-1} &= \sum_{j\geq 1} q^{(2n-1+\alpha)j+1} \sum_{0\leq 2k\leq j-2} q^{2k^2+2k} \begin{bmatrix} j-1 \\ 2k+1 \end{bmatrix}_q \\ &= \sum_{j,k\geq 0} q^{(2n-1+\alpha)(j+2k+2)+1+2k^2+2k} \begin{bmatrix} j+2k+1 \\ 2k+1 \end{bmatrix}_q \\ &= \sum_{k\geq 0} q^{(2n-1+\alpha)(2k+2)+1+2k^2+2k} \sum_{j\geq 0} q^{(2n-1+\alpha)j} \begin{bmatrix} j+2k+1 \\ 2k+1 \end{bmatrix}_q \\ &= \sum_{k\geq 0} \frac{q^{(2n-1+\alpha)(2k+2)+1+2k^2+2k}}{(q^{2n-1+\alpha};q)_{2k+2}}. \end{aligned}$$

For $\alpha = 0$ we have

$$\sum_{k\geq 0} \frac{q^{4nk+4n-1+2k^2}}{(q^{2n-1};q)_{2k+2}} = \sum_{j\geq 0} \frac{q^{2j^2+4nj+4n-1}}{(q^{2n-1};q^2)_{j+1}(q^{2n};q^2)_{j+1}},$$

and for $\alpha = 1$

$$\sum_{k\ge 0}\frac{q^{4nk+4n+1+2k^2+2k}}{(q^{2n};q)_{2k+2}}=\sum_{j\ge 0}\frac{q^{2j^2+2j+4nj+4n+1}}{(q^{2n+1};q^2)_{j+1}(q^{2n};q^2)_{j+1}},$$

as it should.

(For other values of α it does not work!)

Finally we will establish that the Al-Salam/Ismail sequences $U_n(\alpha)/(q;q^2)_n$ converge about twice as fast to S_∞ resp. T_∞ as S_n resp. T_n.

For that we need two simple facts:

$$\begin{aligned}\begin{bmatrix}A\\B\end{bmatrix}_q&=\frac{(1-q^A)(1-q^{A-1})\dots(1-q^{A-B+1})}{(q;q)_B}\\&=\frac{1}{(q;q)_B}\Big(1-q^{A-B+1}+O(q^{A-B+2})\Big)\end{aligned}$$

and

$$\frac{1}{(q^c;q)_n}=1+q^c+O(q^{c+1}).$$

Therefore

$$\begin{aligned}S_n=\sum_{j\ge 0}\begin{bmatrix}n\\2j\end{bmatrix}_q q^{2j^2}&=1+\sum_{j\ge 1}\frac{q^{2j^2}}{(q;q)_{2j}}\Big(1-q^{n-2j+1}+O(q^{n-2j+2})\Big)\\&=\sum_{j\ge 0}\frac{q^{2j^2}}{(q;q)_{2j}}-q^{n+1}+O(q^{n+2}).\end{aligned}$$

Similarly

$$\begin{aligned}T_n=\sum_{j\ge 0}\begin{bmatrix}n\\2j+1\end{bmatrix}_q q^{2j^2+2j}&=\frac{1-q^n}{1-q}+\sum_{j\ge 1}\frac{q^{2j^2+2j}}{(q;q)_{2j+1}}\Big(1-q^{n-2j}+O(q^{n-2j+1})\Big)\\&=\sum_{j\ge 0}\frac{q^{2j^2+2j}}{(q;q)_{2j+1}}-q^n+O(q^{n+1}).\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{U_n(0)}{(q;q^2)_n}&=\frac{1}{(q;q^2)_n}\sum_{j\ge 0}\begin{bmatrix}n-j\\j\end{bmatrix}_{q^2}q^{2j^2}(q^{2j+1};q^2)_{n-2j}\\&=\sum_{j\ge 0}\begin{bmatrix}n-j\\j\end{bmatrix}_{q^2}q^{2j^2}\frac{1}{(q;q^2)_j(q^{2n-2j+1};q^2)_j}\\&=1+\sum_{j\ge 1}\frac{1}{(q^2;q^2)_j}\Big(1-q^{2n-4j+2}+O(q^{2n-4j+4})\Big)\frac{q^{2j^2}}{(q;q^2)_j}\\&\qquad\qquad\times\Big(1+q^{2n-2j+1}+O(q^{2n-2j+3})\Big)\end{aligned}$$

$$= \sum_{j\geq 0} \frac{q^{2j^2}}{(q;q)_{2j}} - q^{2n} + O(q^{2n+1}).$$

Similarly,

$$\begin{aligned}
\frac{U_n(1)}{(q;q^2)_n} &= \frac{1}{(q;q^2)_n} \sum_{j\geq 0} \begin{bmatrix} n-j \\ j \end{bmatrix}_{q^2} q^{2j^2+2j} (q^{2j+3};q^2)_{n-2j} \\
&= \frac{1-q^{2n+1}}{1-q} + \sum_{j\geq 1} \begin{bmatrix} n-j \\ j \end{bmatrix}_{q^2} q^{2j^2+2j} \frac{1}{(q;q^2)_{j+1}(q^{2n-2j+3};q^2)_{j-1}} \\
&= \frac{1-q^{2n+1}}{1-q} + \sum_{j\geq 1} \frac{1}{(q^2;q^2)_j} \Big(1 - q^{2n-4j+2} + O(q^{2n-4j+4})\Big) \frac{q^{2j^2+2j}}{(q;q^2)_{j+1}} \\
&\qquad\qquad \times \Big(1 + q^{2n-2j+3} + O(q^{2n-2j+5})\Big) \\
&= \sum_{j\geq 0} \frac{q^{2j^2+2j}}{(q;q)_{2j+1}} - q^{2n+1} + O(q^{2n+2}).
\end{aligned}$$

3. Identities of Garrett/Ismail/Stanton type

In [8] Garrett et al. presented a new parameterized generalization of the celebrated Rogers–Ramanujan identities. As a by-product of an Engel study, Andrews et al. [5] derived a polynomial version of it which in the limit coincides with the Garrett/Ismail/Stanton result. In [9] Ismail et al. have put the polynomial version from [5] into the context of orthogonal polynomials, in particular, of the Al-Salam/Ismail polynomials U_n.

It turned out that not only the Rogers–Ramanujan identities but also other entries listed by Slater [16] give rise to this type of generalization. For instance, in [9] it was shown that for $m \geq 0$,

$$(-1)^m q^{m^2+m} U_n(m+1) = U_m(0)\, U_{m+n}(1) - U_{m-1}(1)\, U_{m+n+1}(0). \tag{8}$$

Above we have proved that

$$\frac{U_\infty(0)}{(q;q^2)_\infty} = S_\infty \quad \text{and} \quad \frac{U_\infty(1)}{(q;q^2)_\infty} = T_\infty. \tag{9}$$

So after dividing both sides of (8) by $(q;q^2)_\infty$ we obtain in the limit $n \to \infty$ an identity of Garrett/Ismail/Stanton type; namely for $m \geq 0$,

$$(-1)^m q^{m^2+m} \sum_{k\geq 0} \frac{q^{2k^2+2(m+1)k}}{(q;q)_{2k+1}(q^{2k+3};q^2)_m} = U_m(0)\, T_\infty - U_{m-1}(1)\, S_\infty. \tag{10}$$

The experimental use of `Engel`, a computer algebra implementation of the q–Engel expansion algorithm, led Andrews et al. [6] to the discovery of an identity similar to

(10) but using Santos polynomials instead; namely for $m \geq 0$,

$$q^m (q;q^2)_m \sum_{k\geq 0} \frac{q^{2k^2+2(m+1)k}}{(q;q)_{2k+1}} = S_m\, T_\infty - T_m\, S_\infty. \tag{11}$$

The corresponding polynomial version, the counterpart to (8), reads as follows. For $m \geq 0$,

$$q^m (q;q^2)_m \sum_{k\geq 0} \begin{bmatrix} n \\ 2k+1 \end{bmatrix}_q q^{2k^2+2(m+1)k} = S_m\, T_{m+n} - T_m\, S_{m+n}. \tag{12}$$

Once found, such identities most often find quite elementary proofs. Nevertheless, the theme of this section is to sketch a framework that helps to explain and to derive identities of this type. We also stress the fact that this approach can be ideally supplemented by computer algebra packages like `Engel` and q–versions of Zeilberger's ("fast") algorithm, as for instance `qZeil` [13]. Such packages not only can help in proving, but also in *finding* such identities.

All what follows is motivated by techniques from orthogonal polynomials as used e. g. in [9]. However, we will not enter this theory (e. g., numerator or associated polynomials) but rather restrict ourselves to recall a few basic facts from the general theory of difference equations.

Let $\mathbb{F}$ be a suitable field, as e. g. $\mathbb{F} = \mathbb{C}(q)$ where q is an indeterminate. Let $\mathbb{F}^\star$ denote the non-zero elements of $\mathbb{F}$. Let us fix two coefficient sequences $\alpha = (\alpha_n)_{n\geq 0}$ and $\beta = (\beta_n)_{n\geq 0}$ with elements in $\mathbb{F}^\star$. Consider the recurrence equation

$$x_n = \alpha_{n-1} x_{n-1} + \beta_{n-2} x_{n-2} \quad (n \geq 2). \tag{13}$$

Suppose the $\mathbb{F}$–sequences $a = (a_n)_{n\geq 0}$ and $b = (b_n)_{n\geq 0}$ are solutions of (13). For $n \geq 0$ we define the discrete Wronskian as usual by

$$W_n(a,b) := \begin{vmatrix} a_n & b_n \\ a_{n+1} & b_{n+1} \end{vmatrix}.$$

As a matter of fact, the solutions a and b are linearly independent over $\mathbb{F}$ if and only if $W_n(a,b) \neq 0$ for all $n \geq 0$. But this is equivalent to $W_0(a,b) \neq 0$ since

$$W_n(a,b) = (-1)^n \beta_{n-1}\beta_{n-2} \dots \beta_0\, W_0(a,b) \quad (n \geq 0).$$

Combining these facts one can easily prove the following theorem.

Theorem 4. *Let α, β, a, and b be sequences as above where a and b are linearly independent solutions of* (13). *Let m be a non-negative integer and let $c(m) = \big(c_n(m)\big)_{n\geq 0}$ satisfy*

$$z_n = \alpha_{m+n-1}\, z_{n-1} + \beta_{m+n-2}\, z_{n-2} \quad (n \geq 2). \tag{14}$$

Then

$$c_n(m) = u_m\, a_{m+n} + v_m\, b_{m+n} \quad (n \geq 0)$$

where

$$u_m = -\frac{c_1(m)\, b_m - c_0(m)\, b_{m+1}}{W_m(a,b)}$$

and

$$v_m = \frac{c_1(m)\, a_m - c_0(m)\, a_{m+1}}{W_m(a,b)}.$$

Now we apply Theorem 4 in order to prove the polynomial identities (8) and (11).

Example 1. Let m be a non-negative integer. The Al-Salam/Ismail polynomials $U_n(m)$ satisfy (14) with $\alpha_n = 1 - q^n$ and $\beta_n = q^n$. So in view of Theorem 4 we can take $a = (a_n)_{n\geq 0}$ with $a_n = U_n(0)$ and $c(m) = \big(c_n(m)\big)_{n\geq 0}$ with $c_n(m) = U_n(m)$. But we also need a linearly independent solution $b = (b_n)_{n\geq 0}$. Let us try $b_n = U_{n-1}(1)$ since then a and b are both solutions of (13). It turns out that b chosen this way is also linearly independent since $W_0(a,b) = U_0(0)U_0(1) - U_1(0)U_{-1}(1) = 1$; therefore we can invoke Theorem 4. It is easily checked that

$$W_m(a,b) = (-1)^m q^{2m} q^{2m-2} \dots q^2 W_0(a,b) = (-1)^m q^{m^2+m},$$

and

$$c_0(m) = U_0(m) = 1 \quad \text{and} \quad c_1(m) = U_1(m) = 1 - q^{2m+1}.$$

For the coefficients we obtain

$$u_m = -(-1)^m q^{-m^2-m}\big((1-q^{2m+1})U_{m-1}(1) - U_m(1)\big) = (-1)^m q^{-m^2+m} U_{m-2}(1)$$

where the last equation is by (14), and similarly,

$$v_m = (-1)^m q^{-m^2-m}\big((1-q^{2m+1})U_m(0) - U_{m+1}(0)\big) = -(-1)^m q^{-m^2+m} U_{m-1}(0).$$

Now Theorem 4 yields identity (11) with m replaced by $m-1$.

Example 2. As an easy consequence of the mixed recurrences for the Santos polynomials S_n and T_n one immediately derives that both polynomials are solutions of

$$x_n = (1+q)\, x_{n-1} - q(1-q^{2n-3})\, x_{n-2} \quad (n \geq 2). \tag{15}$$

In view of (13) we have $\alpha_n = 1+q$ and $\beta_n = -q(1-q^{2n+1})$. Setting $a_n = S_n$ and $b_n = T_n$ we see that the corresponding sequences a and b are linearly independent since $W_0(a,b) = S_0T_1 - T_0S_1 = 1$. In addition, for any non-negative integer m the sequence $c(m) = \big(c_n(m)\big)_{n\geq 0}$ with

$$c_n(m) = \sum_{k\geq 0} \begin{bmatrix} n \\ 2k+1 \end{bmatrix}_q q^{2k^2+2(m+1)k}$$

satisfies

$$z_n = (1+q)\, z_{n-1} - q(1-q^{2(m+n)-3})\, z_{n-2} \quad (n \geq 2). \tag{16}$$

This e. g. can be proven automatically with the package `qZeil`. Summarizing, we are in the position to invoke Theorem 4. It is easily checked that

$$W_m(a,b) = q^m(1-q^{2m-1})(1-q^{2m-3})\dots(1-q)W_0(a,b) = q^m(q;q^2)_m,$$

and

$$c_0(m) = 0 \quad \text{and} \quad c_1(m) = 1.$$

Hence by Theorem 4, identity (8) is proved.

Finally we demonstrate the applicability of Theorem 4 by deriving a related identity which to our knowledge is new.

Example 3. By using the package `qZeil` one finds that

$$c_n(m) = \sum_{k \ge 0} \begin{bmatrix} n \\ 2k \end{bmatrix}_q q^{2k^2+2mk}$$

is also a solution of (16); this heuristic procedure will be described in a forthcoming paper. Thus we can take a and b as in Example 2 which gives the same expression for $W_m(a, b)$. The only difference to the previous situation is that now

$$c_0(m) = 1 \quad \text{and} \quad c_1(m) = 1.$$

We obtain for the coefficients

$$u_m = -\frac{q^{-m}}{(q; q^2)_m}(T_m - T_{m+1}) = \frac{1}{(q; q^2)_m} S_m$$

where the last equation is by the mixed Santos recurrence, and similarly,

$$v_m = \frac{q^{-m}}{(q; q^2)_m}(S_m - S_{m+1}) = -\frac{q}{(q; q^2)_m} T_m.$$

Theorem 1 now implies that for $m \ge 0$,

$$(q; q^2)_m \sum_{k \ge 0} \begin{bmatrix} n \\ 2k \end{bmatrix}_q q^{2k^2+2mk} = S_m\, S_{m+n} - q\, T_m\, T_{m+n}. \tag{17}$$

The corresponding limiting version, i. e. for $n \to \infty$, reads as follows. For $m \ge 0$,

$$(q; q^2)_m \sum_{k \ge 0} \frac{q^{2k^2+2mk}}{(q; q)_{2k}} = S_m\, S_\infty - q\, T_m\, T_\infty. \tag{18}$$

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