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ON THE NUMBER OF PARTITIONS OF $\{1, \ldots, n\}$ INTO TWO SETS OF EQUAL CARDINALITIES AND EQUAL SUMS

BY
HELMUT PRODINGER

ABSTRACT. Let $A(n)$ be the number of partitions of $\{1, \ldots, n\}$ into two sets A, B of cardinality $n/2$ such that $\sum_{k\in A} k = \sum_{k\in B} k$. Then there is the asymptotic result

$$A(n) \sim \frac{2^n}{n^2} \frac{4\sqrt{3}}{\pi} \quad \text{as} \quad n \to \infty, \qquad n \equiv 0 \pmod 4.$$

1. Introduction. Suppose that the best n tennis players play a master tournament in such a way that, as a first step, two sets of $n/2$ players and equal power play two sub-tournaments.

In mathematical language this reads: The set $\{1, \ldots, n\}$ is partitioned into two sets A, B of cardinality $n/2$ such that

$$\sum_{k\in A} k = \sum_{k\in B} k = \frac{n(n+1)}{4}. \tag{1}$$

In this paper the number $A(n)$ of such partitions is considered. Apparently $n \equiv 0 \pmod 4$ must hold. For instance, for $n = 4$ there are two solutions $A = \{1, 4\}$, $B = \{2, 3\}$ and $A = \{2, 3\}$, $B = \{1, 4\}$, hence $A(4) = 2$.

An asymptotic answer is

THEOREM.

$$A(n) = \left|\left\{(\varepsilon_1, \ldots, \varepsilon_n) \,\middle|\, \varepsilon_i \in \{-1, 1\}, \sum_{k=1}^{n} \varepsilon_k = 0, \sum_{k=1}^{n} \varepsilon_k k = 0\right\}\right| \sim \frac{2^n}{n^2} \frac{4\sqrt{3}}{\pi}.$$

The proof of this result is along the lines of [1], where it is shown that

$$B(n) = \left|\left\{(\varepsilon_{-n}, \ldots, \varepsilon_n) \,\middle|\, \varepsilon_i \in \{0, 1\}, \sum_{k=-n}^{n} \varepsilon_k k = 0\right\}\right| \sim \frac{2^{2n+1}}{n^{3/2}} \sqrt{\frac{3}{\pi}}.$$

However, the present situation is more complicated.

2. Proof of the Theorem. $A(n)$ is the constant term in the expansion of $\prod_{k=1}^{n} (uz^k + u^{-1}z^{-k})$. This yields with $u = e^{ix}$, $z = e^{iy}$

$$\prod_{k=1}^{n} (uz^k + u^{-1}z^{-k}) = 2^n \prod_{k=1}^{n} \cos(x + ky) =: 2^n f_n(x, y).$$

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Note that

$$f_n(\pi+x, y)=f_n(x, y) \quad \text{if} \quad n\equiv 0, 2 \pmod 4$$

and

$$f_n(x, \pi+y)=f_n(x, y) \quad \text{if} \quad n\equiv 0, 3 \pmod 4.$$

Since $f_n(x, y)$ is just a trigonometrical polynomial, its constant term (which is $2^{-n}A(n)$) is found by integrating. Hence

$$\begin{aligned} 4\pi^2 2^{-n}A(n) &= \int_{-\pi/2}^{3\pi/2}\int_{-\pi/2}^{3\pi/2} f_n(x, y)\,dx\,dy \\ &= \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} \{f_n(x, y)+f_n(\pi+x, y)+f_n(x, \pi+y) \\ &\quad +f_n(\pi+x, \pi+y)\}\,dx\,dy \\ &= \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} 4f_n(x, y)\,dx\,dy, \quad \text{only if} \quad n\equiv 0 \pmod 4. \end{aligned}$$

Hence the condition $n\equiv 0 \pmod 4$ is assumed to hold throughout the rest of this paper.

Now the integrand will be estimated for values of y not near to the origin.

$$\begin{aligned} \left[\prod_{k=1}^{n}\cos(x+ky)\right]^2 &= \prod_{k=1}^{n}(1-\sin^2(x+ky)) \\ &< \exp\left[-\sum_{k=1}^{n}\sin^2(x+ky)\right] \qquad (2) \\ &= \exp\left[-\frac{n}{2}+\frac{\cos((n+1)y+2x)\cdot\sin ny}{2\sin y}\right]=0(e^{-\beta n}), \end{aligned}$$

with $\beta>0$ for $\pi/2(n+1)\le|y|\le\pi/2$. Hence the integration with respect to y is only to be done in the interval $[-\pi/2(n+1), \pi/2(n+1)]$:

$$\pi^2 2^{-n}A(n)\sim\int_{-\pi/2(n+1)}^{\pi/2(n+1)}\int_{-\pi/2}^{\pi/2} f_n(x, y)\,dx\,dy.$$

Now assume that $|(n+1)y+2x|\ge\pi/2$, $|y|\le\pi/2(n+1)$, $|x|\le\pi/2$ holds. Since $f_n(-x, -y)=f_n(x, y)$, it is sufficient to discuss

$$(n+1)y+2x\ge\frac{\pi}{2}, \qquad y\le\frac{\pi}{2(n+1)}, \qquad x\le\frac{\pi}{2}.$$

In the estimation (2), the cosine is negative and thus the integrand is again $0(e^{-\gamma n})$, $\gamma>0$. Hence

$$\pi^2 2^{-n}A(n)\sim\iint f_n(x, y)\,dx\,dy,$$

where the integration is to be done in the domain

$$D=\left\{(x, y)\ \Big|\ |x|\le\frac{\pi}{2}, |y|\le\frac{\pi}{2(n+1)}, |(n+1)y+2x|\le\frac{\pi}{2}\right\}.$$

Note that $-\pi/2\le x+ky\le\pi/2$ holds for $k=1,\dots,n$ in this domain. Also note that $\cos z\le\exp(-\frac{1}{2}z^2)$ for $-\pi/2\le z\le\pi/2$. Thus

$$\begin{aligned}\pi^2 2^{-n}A(n)&\sim\iint_D f_n(x, y)\,dx\,dy\le\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\tfrac{1}{2}\sum_{k=1}^{n}(x+ky)^2\right]dx\,dy\\&\sim\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\tfrac{1}{2}\left(nx^2+n^2xy+\frac{n^2}{3}y^2\right)\right]dx\,dy\\&=\frac{1}{\sqrt{n}}\frac{\sqrt{3}}{n^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-(x^2+2rxy+y^2)/2)\,dx\,dy\\&=\frac{\sqrt{3}}{n^2}\cdot 4\pi\qquad(\text{with } r=\sqrt{3}/2).\end{aligned}$$

Now for $|x|<n^{-1/3}$, $|y|<n^{-4/3}$:

$$\begin{aligned}\prod_{k=1}^{n}\cos(x+ky)&=\prod_{k=1}^{n}\exp(-\tfrac{1}{2}(x+ky)^2)\prod_{k=1}^{n}\{1+0((x+ky)^4)\}\\&=\exp\left[-\tfrac{1}{2}\sum_{k=1}^{n}(x+ky)^2+0(n^{-1/3})\right].\end{aligned}$$

Note that for $(x, y)\in D$ the integrand is positive. Hence

$$\begin{aligned}\iint_D f_n(x, y)\,dx\,dy&>\int_{-n^{-4/3}}^{n^{-4/3}}\int_{-n^{-1/3}}^{n^{-1/3}} f_n(x, y)\,dx\,dy\\&\sim\int_{-n^{-4/3}}^{n^{-4/3}}\int_{-n^{-1/3}}^{n^{-1/3}}\exp\left[-\tfrac{1}{2}\sum_{k=1}^{n}(x+ky)^2\right]dx\,dy\sim\frac{\sqrt{3}}{n^2}4\pi.\end{aligned}$$

Therefore $(\sqrt{3}\cdot 4\pi)/n^2$ is an asymptotic equivalent for $\pi^2 2^{-n}A(n)$.

3. Miscellaneous. Here are some numerical values.

n	$A(n)$	$\frac{2^n 4\sqrt{3}}{n^2\pi}$	$A(n)\Big/\frac{2^n 4\sqrt{3}}{n^2\pi}$
4	2	2.205316	0.9069
8	8	8.821264	0.9069
12	58	62.72899	0.9246
16	526	564.5609	0.9317
20	5448	5781.104	0.9424
24	61108	64234.48	0.9513
28	723354	755082.9	0.9580

In the language of probability theory, the theorem can be reformulated. If ε_k are independent identically distributed random variables with values -1 and 1, each with probability $\frac{1}{2}$, then

$$P\left(\sum_{k=1}^{n} \varepsilon_k k = 0 \quad \text{and} \quad \sum_{k=1}^{n} \varepsilon_k = 0\right) \sim \frac{1}{n^2}\frac{4\sqrt{3}}{\pi}. \tag{3}$$

Now

$$P\left(\sum_{k=1}^{n} \varepsilon_k = 0\right) = \binom{n}{n/2} 2^{-n} \sim \sqrt{\frac{2}{\pi n}}, \tag{4}$$

by Stirling's formula. Furthermore,

$$P\left(\sum_{k=1}^{n} \varepsilon_k k = 0\right) \sim \frac{1}{n^{3/2}} \sqrt{\frac{6}{\pi}}, \tag{5}$$

which can be derived by van Lint's method [1].

It is worth noting that $f_n(x, y)$ is the characteristic function of the random vector $S = X_1 + \cdots + X_n$ where $X_k = \pm(1, k)$ each with probability $\frac{1}{2}$. From the Liapounov Central Limit Theorem it follows that if $S = (S_1, S_2)$ then $(S_1 n^{-1/2}, \sqrt{3} S_2 n^{-3/2})$ is asymptotically normally distributed with the density function

$$\frac{1}{2\pi\sqrt{1-r^2}} \exp\{-(t^2 - 2rtu + u^2)/2(1-r^2)\},$$

where $r = \sqrt{3}/2$ is the asymptotic correlation (see [2]).

Thus (3) is not unexpected; (3) is two times the product of (4) and (5) and this factor 2 is just $(1-r^2)^{-1/2}$.

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Institut für Mathematische Logik
und Formale Sprachen, TU Wien
1040 Wien Gußhausstraße 27–29 Austria