6. COMMENTS

Even though each Fermat or Mersenne number is not the power (greater than one) of an integer, it is not known whether they are square-free. Naturally, we make a similar conjecture.

CONJECTURE 2: For each prime p and positive integer i, the number $L(p^i)$ is square-free.

REMARK: It has been shown in [5] that the congruence $2^{p-1} \equiv 1 \pmod{p^2}$ is closely related to the square-freeness of the Fermat and Mersenne numbers. We have shown, by a similar method, that this is also the case for the numbers $L(p^i)$.

It is well known that $(p,\ 2^p - 1) = 1$ and $(n,\ 1 + 2^{2^n}) = 1$. Since the prime divisors of $L(p^i)$ are of the form $1 + kp^{i+1}$ [4, p. 106], it follows that

$$(i,\ L(p^i)) = 1.$$

Finally, we see that while $L(p^i)$ possesses many interesting properties, there remain unanswered some very elementary questions about this class of numbers.

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FIBONACCI NUMBERS OF GRAPHS

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1. INTRODUCTION

According to [1, p. 45], the total number of subsets of $\{1, \ldots, n\}$ such that no two elements are adjacent is F_{n+1}, where F_n is the nth Fibonacci number, which is defined by

$$F_0 = F_1 = 1,\ F_n = F_{n-1} + F_{n-2}.$$

The sequence $\{1, \ldots, n\}$ can be regarded as the vertex set of the graph P_n in Figure 1. Thus, it is natural to define the Fibonacci number $f(X)$ of a (simple) graph X with vertex set V and edge set E to be the total number of subsets S of V such that any two vertices of S are not adjacent.

The Fibonacci number of a graph X is the same as the number of complete (induced) subgraphs of the complement graph of X. (Our terminology covers the empty graph also.)

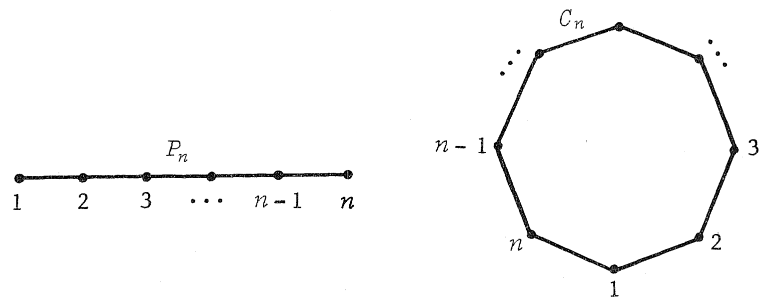

Fig. 1

In [1, p. 46] the case of a cycle C_n with n vertices is considered, as in Figure 1. The Fibonacci number $f(C_n)$ of such a cycle equals the nth Lucas number F_n^*, defined by

$$F_0^* = 2,\ F_1^* = 1,\ F_n^* = F_{n-1}^* + F_{n-2}^*.$$

Let $X_1 = (V, E_1)$ and $X_2 = (V, E_2)$ be two graphs with $E_1 \subseteq E_2$, then

$$f(X_1) \geq f(X_2).$$

So the following simple estimation results:

$$n + 1 = f(K_n) \leq f(X) \leq f(\overline{K_n}) = 2^n, \tag{1.1}$$

where X is a graph with n vertices, and K_n is the complete graph with n vertices and $\overline{K_n}$ its complement.

If X, Y are disjoint graphs, then we trivially obtain, for the Fibonacci number of the union $X \cup Y$,

$$f(X \cup Y) = f(X) \cdot f(Y).$$

2. THE FIBONACCI NUMBERS OF TREES

Trivially, the graph P_n is a tree with $f(P_n) = F_{n+1}$. Another simple example for a tree is the star S_n:

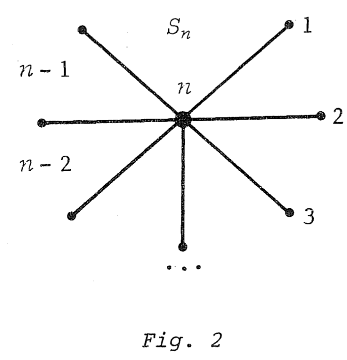

Fig. 2

The Fibonacci number $f(S_n)$ can be computed by counting the number of admissible vertex subsets (they do not contain two adjacent vertices) containing the vertex n or not containing n. Thus

$$f(S_n) = 1 + 2^{n-1}.$$

THEOREM 2.1: Let X be a tree with n vertices, then

$$F_{n+1} \leq f(X) \leq 2^{n-1} + 1.$$

PROOF: First, we prove the second inequality by induction. For $n = 1, 2$, it is trivial. Let X be a tree with $n + 1$ vertices and let v be an endpoint of X. The Fibonacci number $f(X)$ can be computed by counting the number of admissible vertex subsets containing v or not containing v. The number of admissible subsets containing v can trivially be estimated by 2^{n-1} and the number of admissible subsets not containing v can be estimated by $2^{n-1} + 1$ using the induction hypothesis. So we obtain

$$f(X) \leq 2^{n-1} + (2^{n-1} + 1) = 2^n + 1.$$

To prove the first inequality, it is necessary to prove a more general form; hence, we assume X to be a forest. We use induction and, for $n = 1, 2$, the estimation is trivial. Now we proceed by the same argument as above. Let $X = (V, E)$ be a forest with $n + 1$ vertices and v be an endpoint of X. Let X_1 be the induced subgraph of the set $V - \{v\}$ and let w be the adjacent vertex of v. Then X_2 denotes the induced subgraph of the set $V - \{v, w\}$. Trivially, X_1 and X_2 are forests with n and $n - 1$ vertices, respectively. By the induction hypothesis, we obtain

$$f(X) = f(X_1) + f(X_2) \geq F_{n+1} + F_n = F_{n+2},$$

and so the theorem is proved.

REMARK 2.2: There are natural numbers m such that no tree X exists with $f(X) = m$. This is evident because natural numbers m exist not contained in intervals of the form $[F_n, 2^{n-1} + 1]$. Further, there are numbers m contained in such intervals that are not Fibonacci numbers of trees.

EXAMPLE 2.3: Let R_n be the graph with $2n$ vertices as in Figure 3.

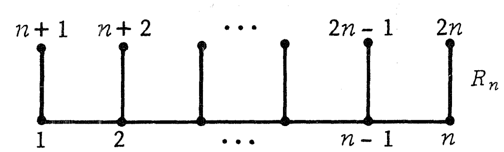

Fig. 3

For the Fibonacci numbers of R_n, we obtain the following recursion

$$f(R_{n+1}) - 2f(R_n) - 2f(R_{n-1}) = 0, \; f(R_1) = 3, \; f(R_2) = 8.$$

The solution of this recursion is

$$f(R_n) = \frac{3 + 2\sqrt{3}}{6}(1 + \sqrt{3})^n + \frac{3 - 2\sqrt{3}}{6}(1 - \sqrt{3})^n.$$

Some other examples are treated in more detail in Section 3.

3. EXAMPLES

Let $X = (V, E)$ be a graph and $y_1, \ldots, y_s$ vertices not contained in V. Then, $Y = (V_1, E_1)$ denotes the graph with

$$V_1 = V \cup \{y_1, \ldots, y_s\} \quad \text{and} \quad E_1 = E \cup \Big\{\{y_1, v_j\} \Big| 1 \leq i \leq s, \; v \in V\Big\}.$$

By the usual recursion argument, we obtain

$$(3.1) \qquad f(Y) = f(X) + 2^s - 1.$$

For an example, we take the following graphs:

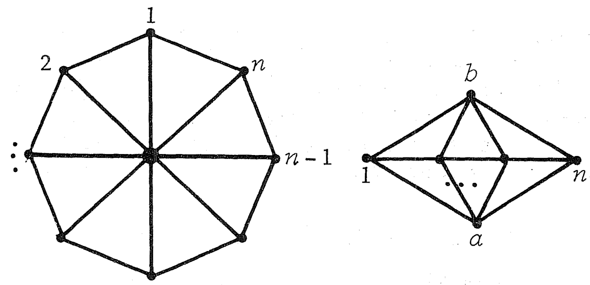

Fig. 4

EXAMPLE 3.2: We consider the graphs Q_n with $2n$ vertices and Q'_n with $2n - 1$ vertices as in Figure 5.

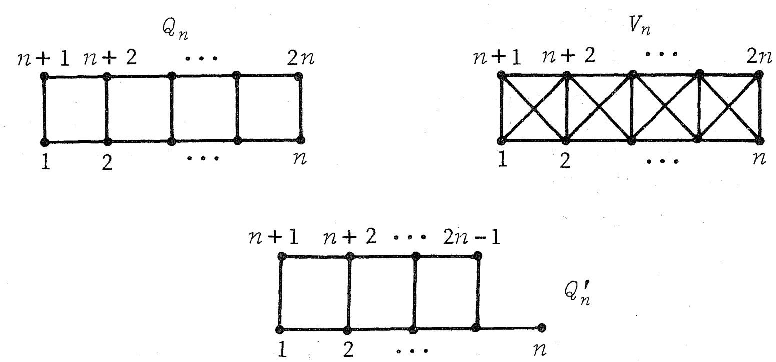

Fig. 5

Let a_n and b_n denote the Fibonacci numbers of Q_n and Q'_n, respectively. By our usual recursion argument, we obtain

1. $a_n = b_n + b_{n-1}$, and
2. $b_n = a_{n-1} + b_{n-1}$.

We now have

3. $b_{n-1} = a_{n-2} + b_{n-2}$,

and by adding (2) and (3),

$$b_n + b_{n-1} = a_{n-1} + a_{n-2} + b_{n-1} + b_{n-2},$$

and so

$$a_n = 2a_{n-1} + a_{n-2};\ a_1 = 3,\ a_2 = 7.$$

This recursion has the solution

$$a_n = \frac{1}{2}(1 + \sqrt{2})^{n+1} + (1 - \sqrt{2})^{n+1} = f(Q_n).$$

EXAMPLE 3.3: Now we consider the graph V_n with $2n$ vertices, as in Figure 5. By the usual recursion argument, we obtain and so

$$f(V_n) = f(V_{n-1}) + 2f(V_{n-2});\ f(V_1) = 3,\ f(V_2) = 5,$$

$$f(V_n) = \frac{1}{3}\left(2^{n+2} + (-1)^{n+1}\right).$$

4. PROBLEMS

PROBLEM 4.1: Compute the Fibonacci number $f(L_n)$ of the lattice graph L_n with n^2 vertices in Figure 6.

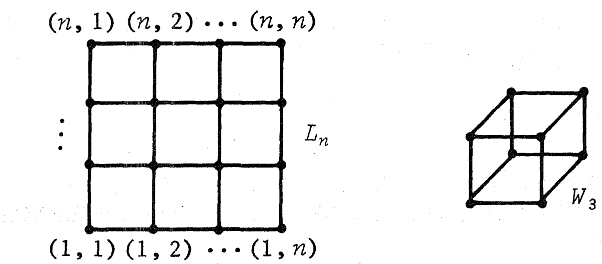

Fig. 6

PROBLEM 4.2: Compute the Fibonacci number of the n-dimensional cube W_n with 2^n vertices in Figure 6.

PROBLEM 4.3: Compute the Fibonacci number of the generalized Peterson graph Pet_n with $4n + 6$ vertices ($n \geq 1$).

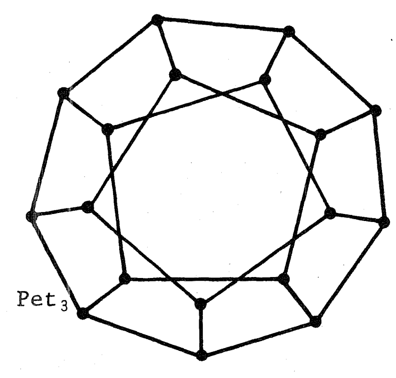

Fig. 7

PROBLEM 4.4: Give a lower bound for $f(X)$ in the case of a planar graph X with n vertices. Give estimations for $f(X)$ if X denotes a regular graph X of degree r or if X denotes an exactly k-connected graph.

PROBLEM 4.5: Let $\omega = (k_n)$ be an increasing sequence of natural numbers, then a sequence Ω of graphs $X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ with $F(X_n) = k_n$ exists such that X_i is embedded as an induced subgraph in X_{i+1}. This is trivial if we take for X_n the complete graph $K_{k_n - 1}$.

We define

$$\delta(\omega) = \inf_{\substack{\Omega = (X_n) \\ f(X_n) = k_n}} \left\{\alpha : |E(X_n)| = O\left(|V(X_n)|^{\alpha}\right)\right\}$$

If γ is a class of increasing sequences of natural numbers (e.g., all increasing sequences or the arithmetic progressions), then we define

$$\Delta(\gamma) = \sup_{\omega \in \gamma} \delta(\omega).$$

Trivially, we obtain $\Delta(\gamma) \leq 2$.

The problem is to give better estimations for $\Delta(\gamma)$ in the general case or in the case where γ is the class of all arithmetic progressions.

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SOME IDENTITIES AND DIVISIBILITY PROPERTIES OF LINEAR SECOND-ORDER RECURSION SEQUENCES

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INTRODUCTION

Following Lucas [5], let P and Q be integers such that

(i) $\quad (P, Q) - 1 \quad$ and $\quad D = P^2 + 4Q \neq 0.$

Let the roots of

(ii) $\quad x^2 = Px + Q$

be

(iii) $\quad a = (P + D^{\frac{1}{2}})/2, \ b = (P - D^{\frac{1}{2}})/2.$

Consider the sequences

(iv) $\quad u^n = (a^n - b^n)/(a - b), \ v_n = a^n + b^n.$

In this article, we examine sums of the form

$$\sum \binom{k}{j} x_n^j (Qx_{n-1})^{k-j} u_j,$$

where $x_n = u_n$ or v_n, and prove that

$$\text{g.c.d. } (u_n, u_{kn}/u_n) \text{ divides } k,$$

and that

$$\text{g.c.d. } (v_n, v_{kn}/v_n) \text{ divides } k \text{ if } k \text{ is odd.}$$

PRELIMINARIES

(1) $(u_n, Q) = (v_n, Q) = 1$
(2) $(u_n, u_{n-1}) = 1$
(3) $D = (a - b)^2$
(4) $P = a + b, \ Q = -ab$
(5) $v_n = u_{n+1} + Qu_{n-1}$
(6) $au_n + Qu_{n-1} = a^n, \ bu_n + Qu_{n-1} = b^n$
(7) $av_n + Qv_{n-1} = a^n(a - b), \ bv_n + Qv_{n-1} = -b^n(a - b)$
(8) $v_n = Pv_{n-1} + Qv_{n-2}$
(9) P even implies v_n even