# SOME POLYNOMIALS RELATED TO THE FIBONACCI POLYNOMIALS

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In my old paper [3], while analyzing an algorithm to construct "Fibonacci partitions of a set", I used the following formula, which is the consequence of an appropriate combinatorial decomposition that will not be repeated here;

$$x^n = \sum_{1 \le k < n} \binom{n-1-k}{k-1} x^k (x-1)^{n-2k} + \sum_{0 \le k < n} \binom{n-1-k}{k} x^{k+1} (x-1)^{n-1-2k}.$$

The polynomials involved are closely related to the *Fibonacci polynomials*

$$\begin{aligned} u_n(z) &= \sum_{0 \le k \le n} \binom{n-k}{k} z^k \\ &= \frac{1}{\sqrt{1+4z}} \left[ \left( \frac{1+\sqrt{1+4z}}{2} \right)^{n+1} - \left( \frac{1-\sqrt{1+4z}}{2} \right)^{n+1} \right]. \end{aligned}$$

Observe that $u_n(1)$ is a Fibonacci number.

Set $z = x/(1-x)^2$, then the formula from above may be rewritten as

$$x^n = x(x-1)^{n-2} u_{n-2}(z) + x(x-1)^{n-1} u_{n-1}(z).$$

To get some expectations and variances, I had to compute

$$\sum_{1 \le k < n} (k)_s \binom{n-1-k}{k-1} x^k (x-1)^{n-2k} + \sum_{0 \le k < n} (k)_s \binom{n-1-k}{k} x^{k+1} (x-1)^{n-1-2k}$$

for $s = 1$ and $s = 2$.

I observed that I could get these values by appropriate differentiation(s) of the explicit form of the Fibonacci polynomials:

$$(x-1)^n z^s \left( \frac{d}{dz} \right)^s z u_{n-2}(z) + x(x-1)^{n-1} z^s \left( \frac{d}{dz} \right)^s u_{n-1}(z).$$

Performing these computations by hand I finally got

$$\sum_{1\le k\le n-1}(-1)^{n+1-k}kx^k \qquad \text{for } s=1$$

and

$$\sum_{2\le k\le n-2}(-1)^{n-k}(n-1-k)k(k-1)x^k \qquad \text{for } s=2.$$

I realized that the computations for higher values of $s$ would be quite messy. While experimenting these days with Maple, I thought again about the old polynomials and conjectured the general formula (for $s\ge 1$):

$$\begin{aligned}&\sum_{1\le k<n}\binom{k}{s}\binom{n-1-k}{k-1}x^k(x-1)^{n-2k}\\&+\sum_{0\le k<n}\binom{k}{s}\binom{n-1-k}{k}x^{k+1}(x-1)^{n-1-2k}\\&=\sum_{0\le k<n}(-1)^{s+n-k}\binom{n-1-k}{s-1}\binom{k}{s}x^k.\end{aligned}$$

In terms of the coefficient of $x^m$ $(0<m<n)$ this means

$$\begin{aligned}&\sum_{1\le k<n}(-1)^k\binom{k}{s}\binom{n-1-k}{k-1}\binom{n-2k}{m-k}\\&+\sum_{0\le k<n}(-1)^k\binom{k}{s}\binom{n-1-k}{k}\binom{n-1-2k}{m-1-k}\\&=\binom{n-1-m}{s-1}\binom{m}{s}(-1)^s.\end{aligned}$$

Then I thought about how to prove this and finally found the following quite elementary computation that I am going to present now.

Since $s\ge 1$, we can assume the same range $1\le k<n$ in both sums. The $k$th term is then

$$\begin{aligned}&(-1)^k\binom{k}{s}\binom{n-1-k}{k-1}\binom{n-2k}{m-k}+(-1)^k\binom{k}{s}\binom{n-1-k}{k}\binom{n-1-2k}{m-1-k}\\&=(-1)^k\binom{k}{s}\binom{n-1-k}{m-1}\binom{m-1}{k-1}+(-1)^k\binom{k}{s}\binom{n-1-k}{m-1}\binom{m-1}{k}\\&=(-1)^k\binom{k}{s}\binom{n-1-k}{m-1}\binom{m}{k}\\&=(-1)^k\binom{m}{s}\binom{m-s}{k-s}\binom{n-1-k}{m-1}.\end{aligned}$$

The sum on $k$ of this expression can be simplified due the equation (5.25) in [1] that I repeat now for convenience:

$$\sum_{k\le l}\binom{l-k}{m}\binom{s}{k-n}(-1)^k=(-1)^{l+m}\binom{s-m-1}{l-m-n},\quad \text{integers } l,m,n\ge 0.$$

It is a disguised form of the Vandermonde convolution.

Therefore

$$\sum_k (-1)^k \binom{m}{s}\binom{m-s}{k-s}\binom{n-1-k}{m-1}$$
$$= \binom{m}{s}(-1)^{n+m}\binom{-s}{n-m-s}$$
$$= \binom{m}{s}(-1)^s\binom{n-m-1}{s-1},$$

and the proof is finished.

What can we learn from it? Computer experiments can be useful; they lead you to conjecture the result. And sometimes an elementary computation does a better job than a more sophisticated one.

## References

1. R.L. Graham, D.E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison Wesley, 1989.
2. H. Prodinger, *On the Number of Fibonacci Partitions of a Set*, The Fibonacci Quarterly **19** (1981), 463–466.
3. H. Prodinger, *Analysis of an Algorithm to Construct Fibonacci Partitions*, RAIRO Informatique Théorique et Applications **18** (1984), 387–394.

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