THE ASYMPTOTIC BEHAVIOR OF THE GOLDEN NUMBERS

Helmut Prodinger

Department of Algebra and Discrete Mathematics
Technical University of Vienna, Austria

In [2] the "Golden polynomials"

$$G_{n+2}(x) = xG_{n+1}(x) + G_n(x), \qquad G_0(x) = -1, G_1(x) = x - 1$$

and their maximal real root g_n (the "golden numbers") were investigated. It was observed that, as $n \to \infty$, $g_n \to \frac{3}{2}$; furthermore it was suggested there might be a more precise formula, since numerical experiments seemed to indicate a dependency on the parity of n of the lower order terms.

This open question will be solved in the present paper.

Solving the recursion for the Golden polynomials by standard methods we get the explicit formula

$$G_n(x) = A\lambda^n + B\mu^n,$$

with

$$\lambda = \frac{x + \sqrt{x^2+4}}{2}, \quad \mu = \frac{x - \sqrt{x^2+4}}{2},$$

$$A = \frac{1}{2\sqrt{x^2+4}}\left(3x - 2 - \sqrt{x^2+4}\right), \quad B = -\frac{1}{2\sqrt{x^2+4}}\left(3x - 2 + \sqrt{x^2+4}\right).$$

Everything is much nicer when we substitute

$$x = u - \frac{1}{u}\ .$$

$G_n(x) = 0$ can be rephrased as $-B/A = (\lambda/\mu)^n$, or

$$\frac{(2u+1)(u-1)}{(u+1)(u-2)} = \left(-u^2\right)^n.$$

Now it is plain to see that, for large n, this equation can only hold if u is either close to 2 or to $u = -\frac{1}{2}$. In both cases, this would mean x is close to $\frac{3}{2}$. Let us assume that u is close to 2. It is clear that the cases when n is even or odd have to be distinguished. We start with $n = 2m$ and rewrite the equation as

$$u - 2 = \frac{(2u+1)(u-1)}{(u+1)} u^{-4m}.$$

We get the asymptotic behavior of the desired solution by a process known as “bootstrapping” which is explained int [1]. First, we set $u = 2 + \delta$, insert $u = 2$ into the right hand side, and get an approximation for δ. Then we insert $u = 2 + \delta$ into the right hand side, expand, and get the next term. This procedure can be repeated to get as many terms as needed. In this way, we get

$$\delta \sim \frac{5}{3} \cdot 16^{-m},$$

and with $u = 2 + \frac{5}{3} \cdot 16^{-m} + \varepsilon$, we find

$$\varepsilon \sim -\frac{25}{6} m \cdot 256^{-m}.$$

From

$$u \sim 2 + \frac{5}{3} \cdot 16^{-m} - \frac{25}{6} m \cdot 256^{-m}$$

we find by substitution

$$x \sim \frac{3}{2} + \frac{25}{12} 16^{-m} - \frac{125}{24} m 256^{-m}.$$

Now let us consider the case n is odd, $n = 2m + 1$. Then our equation is

$$u - 2 = -\frac{(2u+1)(u-1)}{(u+1)u^2} u^{-4m},$$

and we find as above

$$u \sim 2 - \frac{5}{12} 16^{-m} - \frac{25}{72} m 256^{-m},$$

and also

$$x \sim \frac{3}{2} - \frac{25}{48} 16^{-m} - \frac{125}{288} m 256^{-m}.$$

Confining ourselves to two terms we write our findings in a single formula as

$$g_n \sim \frac{3}{2} + (-1)^n \frac{25}{12} 4^{-n},$$

which matches perfectly with the empirical data from [2].

References

1. D.Greene, D.Knuth, *Mathematics for the Analysis of Algorithms*, Birkhäuser, 1981.
2. G.Moore, *The limit of the golden numbers is* 3/2, The Fibonacci Quarterly **32** (1994), 211–217.

AMS classification: 11B39, 11B37, 11C08

TU Vienna
Wiedner Hauptstrasse 8–10
A-1040 Vienna
Austria
E-mail address: proding@email.tuwien.ac.at