SUMS OF POWERS OF FIBONACCI POLYNOMIALS

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Abstract. Using the explicit (Binet) formula for the Fibonacci polynomials, a summation formula for powers of Fibonacci polynomials is derived straightforwardly, which generalizes a recent results for squares that appeared in this journal.

Consider the recursion

$$u(n+1) = xu(n) + u(n-1), \qquad u(0) = 0,\ u(1) = 1;$$

the quantities $u(n) = u(n;x)$ are variants of the *Fibonacci polynomials*.

There is the *Binet form*

$$u(n) = \frac{1}{D}[\alpha^n - \beta^n],$$

with

$$D = \sqrt{x^2+4}, \qquad \alpha = \frac{x+D}{2}, \qquad \beta = \frac{x-D}{2}.$$

Note that $\alpha\beta = -1$.

Let

$$U_d(n) := \sum_{k=0}^{n} u^d(k).$$

For $d = 2$, these sums have been evaluated in [1] using *matrix methods*, which resulted in some quite long calculations. We decided to replace the parameter A from [1] by x to emphasize the nature of (Fibonacci) *polynomials*.

Here, we want to use the Binet form directly and attack the general sum with arbitrary exponent d. After all, all we have to do is to sum the (finite) geometric series! We would like to emphasize that such evaluations are well within the limits of modern computer algebra systems which produce (in principle) such results automatically for any value of d.

Note that

$$u^d(k) = \frac{1}{D^d}\sum_{i=0}^{d}\binom{d}{i}(-1)^{d-i}\big(\alpha^i\beta^{d-i}\big)^k.$$

Therefore

$$U_d(n) = \frac{1}{D^d}\sum_{i=0}^{d}\binom{d}{i}(-1)^{d-i}\sum_{k=0}^{n}\big(\alpha^i\beta^{d-i}\big)^k$$

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$$= \frac{1}{D^d} \sum_{i=0}^{d} \binom{d}{i} (-1)^{d-i} \frac{1 - \left(\alpha^i \beta^{d-i}\right)^{n+1}}{1 - \alpha^i \beta^{d-i}},$$

which is an *explicit* formula.

Notice that in the exceptional case $d = 4d'$ and $i = 2d'$, the sum

$$\sum_{k=0}^{n} \left(\alpha^i \beta^{d-i}\right)^k$$

must be interpreted as $n + 1$.

Perhaps more attractive is the generating function

$$\begin{aligned} F_d(z) &:= \sum_{n \geq 0} z^n U_d(n) \\ &= \frac{1}{D^d} \sum_{i=0}^{d} \binom{d}{i} (-1)^{d-i} \frac{1}{1 - \alpha^i \beta^{d-i}} \left[\frac{1}{1-z} - \frac{\alpha^i \beta^{d-i}}{1 - z\alpha^i \beta^{d-i}} \right]. \end{aligned}$$

(Again, the instance $d = 4d'$ and $i = 2d'$ must be interpreted as a limit.)

Here is a little list; simplification has been done by Maple:

$$\begin{aligned} d = 1 : &\qquad \frac{z}{(1-z)(1-xz-z^2)}, \\ d = 2 : &\qquad \frac{z}{(1+z)(1-(2+x^2)z+z^2)}, \\ d = 3 : &\qquad \frac{z(1-2xz-z^2)}{(1-z)(1-xz(3+x^2)-z^2)(1+xz-z^2)}, \\ d = 4 : &\qquad \frac{z(1+z)(1-(2+3x^2)z+z^2)}{(1-z)^2(1+(2+x^2)z+z^2)(1-2z-x^2(4+x^2)z+z^2)}, \\ d = 5 : &\qquad \frac{z(1-(3x+4x^3)z-(2+8x^2+6x^4)z^2+(3x+4x^3)z^3+z^4)}{(1-z)(1+(3x+x^3)z-z^2)(1-xz-z^2)(1-(5x+5x^3+x^5)z-z^2)}. \end{aligned}$$

Simplifications like $U_2(n) = \frac{u(n)u(n+1)}{x}$ (obtained in [1]) can be checked by Maple.

But it is always possible to get such a representation; let us demonstrate this for $d = 3$:

$$\begin{aligned} F_3(z) &= \frac{3(z+1)}{x(x^2+4)} \frac{1}{(z-\alpha)(z-\beta)} + \frac{2}{x(x^2+3)} \frac{1}{1-z} \\ &\quad - \frac{(x^2+1)(z+1)}{x(x^2+3)(x^2+4)} \frac{1}{(z+\alpha^3)(z+\beta^3)} \\ &= -\frac{3(z+1)}{x(x^2+4)} \frac{1}{(1+z\alpha)(1+z\beta)} + \frac{2}{x(x^2+3)} \frac{1}{1-z} \\ &\quad + \frac{(x^2+1)(z+1)}{x(x^2+3)(x^2+4)} \frac{1}{(1-z\alpha^3)(1-z\beta^3)} \\ &= -\frac{3(z+1)}{x(x^2+4)D} \left[\frac{\alpha}{1+z\alpha} - \frac{\beta}{1+z\beta} \right] + \frac{2}{x(x^2+3)} \frac{1}{1-z} \end{aligned}$$

$$+\frac{(z+1)}{x(x^2+3)(x^2+4)D}\Big[\frac{\alpha^3}{1-z\alpha^3}-\frac{\beta^3}{1-z\beta^3}\Big].$$

Therefore

$$\begin{aligned}U_3(n)=[z^n]F_3(z)&=-\frac{3(-1)^n}{x(x^2+4)D}\Big[\alpha^{n+1}-\beta^{n+1}-\alpha^n+\beta^n\Big]+\frac{2}{x(x^2+3)}\\&\quad+\frac{1}{x(x^2+3)(x^2+4)D}\Big[\alpha^{3(n+1)}-\beta^{3(n+1)}+\alpha^{3n}-\beta^{3n}\Big]\\&=\frac{3(-1)^n}{x(x^2+4)}\Big[u(n+1)-u(n)\Big]+\frac{2}{x(x^2+3)}\\&\quad+\frac{1}{x(x^2+3)(x^2+4)}\Big[u(3n+3)+u(3n)\Big].\end{aligned}$$

Since the instance $d=4$ contains the exceptional case, we will also work it out:

$$\begin{aligned}F_4(z)&=\frac{z+1}{(x^2+4)^2}\frac{1}{(z-\alpha^4)(z-\beta^4)}-\frac{3}{(x^2+4)^2}\frac{1}{1-z}\\&\quad-\frac{4(z+1)}{(x^2+4)^2}\frac{1}{(z+\alpha^2)(z+\beta^2)}+\frac{6}{(x^2+1)^2}\frac{1}{(1-z)^2}\\&=\frac{z+1}{(x^2+4)^2}\frac{1}{(1-z\alpha^4)(1-z\beta^4)}-\frac{3}{(x^2+4)^2}\frac{1}{1-z}\\&\quad-\frac{4(z+1)}{(x^2+4)^2}\frac{1}{(1+z\alpha^2)(1+z\beta^2)}+\frac{6}{(x^2+1)^2}\frac{1}{(1-z)^2}\\&=\frac{z+1}{x(x^2+4)^2(x^2+2)D}\Big[\frac{\alpha^4}{1-z\alpha^4}-\frac{\beta^4}{1-z\beta^4}\Big]-\frac{3}{(x^2+4)^2}\frac{1}{1-z}\\&\quad-\frac{4(z+1)}{(x^2+4)^2D}\Big[\frac{\alpha^2}{1+z\alpha^2}-\frac{\beta^2}{1+z\beta^2}\Big]+\frac{6}{(x^2+1)^2}\frac{1}{(1-z)^2}.\end{aligned}$$

Hence

$$\begin{aligned}U_4(n)=[z^n]F_4(z)&=\frac{1}{x(x^2+4)^2(x^2+2)}\Big[u(4n+4)+u(4n)\Big]-\frac{3}{(x^2+4)^2}\\&\quad-\frac{4(-1)^n}{(x^2+4)^2}\Big[u(2n+2)-u(2n)\Big]+\frac{6}{(x^2+1)^2}(n+1).\end{aligned}$$

And here is the result for $d=5$:

$$\begin{aligned}U_5(n)&=-\frac{2(4+3x^2)}{x(x^2+3)(5+5x^2+x^4)}-\frac{5(-1)^n}{x(x^2+3)(x^2+4)^2}\Big[u(3n+3)-u(3n)\Big]\\&\quad+\frac{1}{x(x^2+4)^2(5+5x^2+x^4)}\Big[u(5n+5)+u(5n)\Big]\\&\quad+\frac{10}{x(x^2+4)^2}\Big[u(n+1)+u(n)\Big].\end{aligned}$$

Similar quantities can be handled in the same style.

References

[1] E. Kilic. Sums of the squares of terms of sequence $\{u_n\}$. *Proc. Indian Acad. Sci. (Math. Sci.)*., 118:27–41, 2008.

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