THE FIBONACCI KILLER

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1. Introduction

We consider the following stochastic process: Assume that a "player" is hit at any time x with probability p. However he dies only after two consecutive hits. We might code this process by $\mathbf{0}$ and $\mathbf{1}$, marking a hit e.g. by a "$\mathbf{1}$". Then the sequences associated with a player can be described by

$$\{\mathbf{0},\mathbf{10}\}^* \cdot \mathbf{11}.$$

The notation $\{\mathbf{0},\mathbf{10}\}^*$ denotes arbitrary sequences consisting of the blocks $\mathbf{0}$ and $\mathbf{10}$, the block $\mathbf{11}$ are the fatal hits. Notice that $\{\mathbf{0},\mathbf{10}\}^*$ are exactly the admissible blocks in the Fibonacci expansion of integers (*Zeckendorf* expansion, cf.[13]). Accordingly, the generating function

$$\frac{p^2z^2}{1-qz-pqz^2} \tag{1.1}$$

has as the coefficient of z^x the probability $\mathbb{P}\{X=x\}$ that the lifetime X of a player is exactly x. The generating function (1.1) is known in the context of the Fibonacci distribution or geometric distribution of order 2, cf. [1], [3], [4], [7], [8], [10], [12].

Here, we are interested in n (independent) players subject to this game and ask when (in the sense of a mean value) the last player dies.

Without the "Fibonacci" restriction, i.e. the maximum of n (independent) geometric random variables, this problem has been studied previously and has some applications. (Compare [5], [11].)

We have obviously

$$\mathbb{P}\Big\{\max\{X_1,\dots,X_n\}\le x\Big\} = \Big(\mathbb{P}\{X\le x\}\Big)^n. \tag{1.2}$$

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The generating function of $\mathbb{P}\{X > x\}$ is given by

$$\frac{1+pz}{1-qz-pqz^2}.$$

We now factor the denominator of this function to obtain

$$1-qz-pqz^2 = (1-az)(1-bz)$$

with

$$a = \frac{q+\sqrt{q^2+4pq}}{2} \quad \text{and} \quad b = \frac{q-\sqrt{q^2+4pq}}{2}.$$

Performing the partial fraction decomposition and extracting coefficients yields

$$\mathbb{P}\{X > x\} = \frac{1}{\sqrt{q^2+4pq}}\left(a^x(a+p) - b^x(b+p)\right).$$

Using (1.2) we obtain the expectation for the maximum lifetime of n players

$$\begin{aligned} \mathbb{E}_n &= \mathbb{E}\max\left\{X_1,\dots,X_n\right\} \\ &= \sum_{x\ge0}\left(1-\left(1-\frac{1}{\sqrt{q^2+4pq}}\left(a^x(a+p)-b^x(b+p)\right)\right)^n\right). \end{aligned} \tag{1.3}$$

By the binomial theorem we obtain

$$\mathbb{E}_n = \sum_{m=1}^{n}(-1)^{m-1}\binom{n}{m}\sum_{x\ge0}\left(Aa^x - Bb^x\right)^m, \tag{1.4}$$

where we use the notation

$$A = \frac{a+p}{\sqrt{q^2+4pq}} = \frac{a^2}{q\sqrt{q^2+4pq}} \quad \text{and} \quad B = \frac{b+p}{\sqrt{q^2+4pq}} = \frac{b^2}{q\sqrt{q^2+4pq}}.$$

For example, in the symmetric case $p = q = \frac{1}{2}$ we have $a = \frac{1+\sqrt{5}}{4}$, $b = \frac{1-\sqrt{5}}{4}$, $A = \frac{5+3\sqrt{5}}{10}$ and $B = \frac{5-3\sqrt{5}}{10}$.

We will find that $\mathbb{E}_n \sim \log_{1/a} n$ and refer for the (technical) proof and a more precise statement to the next section.

2. Asymptotic Analysis

In (1.4) we found the expression

$$\mathbb{E}_n = \sum_{m=1}^{n}(-1)^{m-1}\binom{n}{m}f(m), \tag{2.1}$$

containing the function

$$f(z) = \sum_{x\ge0}\left(Aa^x - Bb^x\right)^z \quad \text{for} \quad \Re z > 0.$$

For an expression of that type we can write a complex contour integral

$$\mathbb{E}_n = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{(-1)^n n!}{z(z-1)\cdots(z-n)} f(z)\,dz, \tag{2.2}$$

where $\mathcal{C}$ is a positively oriented Jordan curve encircling the points $1, 2, \dots, n$ (and no other integer points); this can easily be checked by residue calculus.

The contours of integration

We will use Rice's method to obtain an asymptotic expansion for $\mathbb{E}_n$. For this, we refer e.g. to [2] and [6]. This method is based on a deformation of the contour of integration. For this purpose we need an analytic continuation of the function f to a region containing a half-plane $\Re z > -\varepsilon$ for $\varepsilon > 0$ (we actually give an analytic continuation to the whole complex plane).

Using the notation $C = \frac{B}{A}$ and $d = \frac{b}{a}$ (observe that $|C| < 1$ and $|d| < 1$) we obtain

$$\begin{aligned} f(z) &= A^z \sum_{x\ge 0} a^{xz} (1 - Cd^x)^z = A^z \sum_{x\ge 0} a^{xz} \sum_{\ell\ge 0} (-1)^\ell C^\ell d^{x\ell} \binom{z}{\ell} \\ &= A^z \sum_{\ell\ge 0} (-1)^\ell C^\ell \binom{z}{\ell} \sum_{x\ge 0} (a^z d^\ell)^x \\ &= A^z \sum_{\ell\ge 0} \binom{z}{\ell} \frac{(-1)^\ell C^\ell}{1 - a^z d^\ell}, \end{aligned} \tag{2.3}$$

where the reversion of the order of summation was justified because of the absolute convergence of the sum for $\Re z > 0$. The sum in the last line gives a valid expression for $f(z)$ for every complex number z which is not a solution of any of the equations $1-a^z d^\ell=0$. In the points $z_{\ell,x}=-\ell\frac{\log d}{\log a}+\frac{2x\pi i}{\log a}$ with $\ell=0,1,\ldots$ and $x\in\mathbb{Z}$ there are simple poles with residue

$$A^{z_{\ell,x}}\binom{z_{\ell,x}}{\ell}\frac{(-1)^{\ell-1}C^\ell}{\log a}.$$

In order to be able to deform the contour of integration we need an estimate for $f(z)$ along the vertical line $\Re z=-u$. For this purpose we write

$$f(z)-\frac{A^z}{1-a^z}=\sum_{x\geq 0}A^z a^{xz}\left((1-Cd^x)^z-1\right)$$

and observe the inequality $\left|(1-Cd^x)^z-1\right|\leq\min(2,|z|Cd^x)$. This yields

$$\left|f(z)-\frac{A^z}{1-a^z}\right|\leq A^{-u}\left(\sum_{0\leq x\leq\log|z|}2|a|^{-xu}+|z|\sum_{x>\log|z|}a^{-xu}Cd^x\right)\ll|z|^\alpha \tag{2.4}$$

for $|d|<a^u<1$ and $\alpha=-u\log a$.

We are now ready to start the deformation of the contour of integration: we take $\mathcal{C}'$ as the new contour and write

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{(-1)^n n!}{z(z-1)\cdots(z-n)}f(z)\,dz&=\frac{1}{2\pi i}\oint_{\mathcal{C}'}\frac{(-1)^n n!}{z(z-1)\cdots(z-n)}f(z)\,dz\\ -\sum\operatorname*{Res}_{z=z_i}\frac{(-1)^n n!}{z(z-1)\cdots(z-n)}f(z),&\end{aligned}\tag{2.5}$$

where the summation runs over all poles z_i between the curves $\mathcal{C}$ and $\mathcal{C}'$. Notice that there is a second order pole at 0. Computation of residues yields (with $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$)

$$\begin{aligned}\operatorname*{Res}_{z=0}\frac{(-1)^n n!}{z(z-1)\cdots(z-n)}f(z)&=\frac{1}{\log a}H_n+\frac{\log A}{\log a}-\frac{1}{2}\\ \operatorname*{Res}_{z=\chi_x}\frac{(-1)^n n!}{z(z-1)\cdots(z-n)}f(z)&=\frac{A^{\chi_x}}{\chi_x\ log a}\frac{n!\Gamma(1-\chi_x)}{\Gamma(n+1-\chi_x)}\quad\text{for } x\neq 0,\end{aligned}\tag{2.6}$$

where $\chi_x=\frac{2x\pi i}{\log a}=z_{0,x}$.

Shifting the upper, the lower and the right part of $\mathcal{C}'$ (cf. the figure) to infinity and observing, that the integrals over these parts of the contour vanish then, yields

$$\begin{aligned}\mathbb{E}_n=&\frac{1}{\log\frac{1}{a}}H_n-\frac{\log A}{\log a}+\frac{1}{2}-\sum_{x\in\mathbb{Z}\setminus\{0\}}\frac{A^{\chi_x}}{\chi_x\log a}\frac{n!\Gamma(1-\chi_x)}{\Gamma(n+1-\chi_x)}\\ &-\frac{1}{2\pi i}\int_{-u-i\infty}^{-u+i\infty}\frac{(-1)^n n!}{z(z-1)\cdots(z-n)}f(z)\,dz.\end{aligned}\tag{2.7}$$

We now use the well-known asymptotic expansions $H_n=\log n+\gamma+O(\frac{1}{n})$ and $\dfrac{n!}{\Gamma(n+1-\chi_x)}=n^{\chi_x}\left(1+O(\frac{x^2}{n})\right)$ (by Stirling's formula) to formulate our main result.

Theorem 1. *The expected maximal lifetime $\mathbb{E}_n$ of n independent players each of which has the Fibonacci distribution (or geometric distribution of order 2) fulfils for $n \to \infty$*

$$\mathbb{E}_n = \log_{1/a} n - \frac{\gamma + \log A}{\log a} + \frac{1}{2} - \varphi(\log_{1/a} n) + O(n^{-u}), \tag{2.8}$$

for $0 < u < \min(1, \frac{\log |d|}{\log a})$ and φ denotes a continuous periodic function of period 1 and mean 0 given by the Fourier expansion

$$\varphi(t) = \frac{1}{\log a} \sum_{x \in \mathbb{Z} \setminus \{0\}} A^{\chi_x} \Gamma(-\chi_x) e^{2x\pi i t} = \frac{1}{\log a} \sum_{x \in \mathbb{Z} \setminus \{0\}} \Gamma(-\chi_x) e^{2x\pi i (t - \log_{1/a} A)}, \tag{2.9}$$

which is rapidly convergent due to the exponential decay of the Γ-function along vertical lines. The remainder term is obtained by a trivial estimate of the integral and the (uniform) O-terms in Stirling's formula.

3. Extensions

Here, we briefly sketch the more general case where k consecutive hits are necessary to kill a player. In this case the probability $\mathbb{P}(X = x)$ was derived by Philippou and Muwafi [9] in terms of multinomial coefficients. As described in the introduction there is a bijection to the sequences

$$\{\mathbf{0}, \mathbf{10}, \mathbf{110}, \dots, \mathbf{1}^{k-1}\mathbf{0}\} \cdot \mathbf{1}^k,$$

which yield the probability generating function

$$\frac{p^k z^k}{1 - qz - pqz^2 - \cdots - p^{k-1} q z^k} = \frac{p^k z^k (1 - pz)}{1 - z + qp^k z^{k+1}} \tag{3.1}$$

for the lifetime of a player (cf. [1, pp. 299ff], [3, p. 428], [7, p. 207], [8]). Likewise, the generating function of $\mathbb{P}\{X > x\}$ is given by

$$\frac{1 - p^k z^k}{1 - z + qp^k z^{k+1}} \tag{3.2}$$

Again we factor the polynomial in the denominator

$$1 - qz - pqz^2 - \cdots - p^{k-1} q z^k = (1 - \alpha z)(1 - \alpha_2 z) \cdots (1 - \alpha_k z)$$

with $|\alpha| > |\alpha_2| \geq \cdots \geq |\alpha_k|$ ($\alpha > 0$). Then we have by partial fraction decomposition and extracting coefficients

$$\mathbb{P}\{X > x\} = A\alpha^x + A_2 \alpha_2^x + \cdots + A_k \alpha_k^x \tag{3.3}$$

with

$$A = \frac{\alpha(\alpha - p)}{q((k+1)\alpha - k)}$$

and similar expressions for $A_2, \dots, A_k$.

For the expectation of the maximal lifetime of n players we obtain

$$\mathbb{E}_{n,k} = \mathbb{E}\max\{X_1, \dots, X_n\} = \sum_{m=1}^{n} (-1)^{m-1} \binom{n}{m} g(m)$$

with

$$g(z) = \sum_{\ell \geq 0} \left(A\alpha^\ell + \dots + A_k\alpha_k^\ell\right)^z \quad \text{for } \Re z > 0.$$

For the purpose of analytic continuation of g we consider

$$g(z) - \frac{A^z}{1-\alpha^z}$$

and proceed as in (2.4) to obtain the continuation and a polynomial estimate for $g(z)$ along some vertical line $\Re z = -\varepsilon$ for sufficiently small $\varepsilon > 0$.

We are now ready to perform similar calculations as in Section 2. Thus we obtain

Theorem 2. *The expected maximal lifetime $\mathbb{E}_{n,k}$ of n players each of which has geometric distribution of order k satisfies*

$$\mathbb{E}_{n,k} = \log_{1/\alpha} n - \frac{\gamma + \log A}{\log \alpha} + \frac{1}{2} + \psi\left(\log_{1/\alpha} n\right) + O(n^{-\varepsilon}),$$

for $0 < \varepsilon < \min(1, \frac{\log|\alpha_2|}{\log \alpha})$ and a continuous periodic function ψ of period 1 and mean 0 whose Fourier expansion is given by

$$\psi(t) = \frac{1}{\log \alpha} \sum_{x \in \mathbb{Z}\setminus\{0\}} A^{\chi_x} \Gamma(-\chi_x) e^{2x\pi i t} = \frac{1}{\log \alpha} \sum_{x \in \mathbb{Z}\setminus\{0\}} \Gamma(-\chi_x) e^{2x\pi i (t - \log_{1/\alpha} A)},$$

where $\chi_x = \frac{2x\pi i}{\log \alpha}$.

By *bootstrapping* we find that, for $k \to \infty$,

$$\alpha \sim 1 - qp^k + \dots \quad \text{and} \quad A \sim 1 + kqp^k + \dots .$$

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