FURTHER RESULTS ON A PROBLEM OF KNÖDEL CONCERNING THE ANALYSIS OF BIN-PACKING

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Abstract. Assume that inputs of size $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, or 1 might occur with equal probabilities. If they are put into boxes of size 1 by the strategy *best fit*, there remains some *empty space*. This quantity has been evaluated asymptotically for *n random insertions*.

1. Introduction

W. Knödel [2] has considered *buckets* of size 1 and inputs of size $\frac{1}{3}, \frac{2}{3}, 1$ which occur with equal probabilities ($\frac{1}{3}$ each). The strategy to fill the buckets is *best fit* ; in his instance a certain number of totally full buckets might occur, plus a certain number m of buckets filled with $\frac{2}{3}$. If $m=0$ one bucket filled $\frac{1}{3}$ is also possible.

The random variable of interest is E_n, i.e. the empty space after n random insertions; it is $\frac{m}{3}$ or, eventually, $\frac{2}{3}$. The present author [4] has sharpened the original analysis [2] and has shown

$$E_n \sim \frac{4}{3\sqrt{3}}\sqrt{\frac{n}{\pi}}.$$

Let us call the described problem the *case 3*.

In this note we want to present the analysis of the *case 4*.

However, we want to mention that a generalization is not at all obvious and definitely requires other methods. The *case 4* is much more complex than the *case 3*, but (concerning the occuring states) one-dimensional, so that the generating functions of interest are related to *context-free grammars* and are henceforth *algebraic*. So we can use the method of *Darboux* to estimate the coefficients. As a general reference for this technique we mention [1]. For 5 or more this changes drastically.

We propose here to show:

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Theorem. *Let E_n be the empty space after n random insertions in the case-4-model. Then*

$$E_n \sim \frac{1}{4}\sqrt{\frac{n}{\pi}}.$$

We ran simulations, each based on 200 experiments of n insertions. We report here the ratio of the average empty space (experiment) over the leading term of the average empty space (theory):

$n = 100$	1.231
$n = 500$	1.053
$n = 1000$	1.080
$n = 2000$	0.898
$n = 3000$	1.038
$n = 5000$	0.970
$n = 20000$	1.025
$n = 100000$	1.040

2. The Analysis

The following figure explains the situation quite easily:

Here, q_i stands for i buckets, filled $\frac{3}{4}$, and p_i for i buckets filled $\frac{3}{4}$, plus one bucket, filled $\frac{1}{2}$. The state β stands for one bucket filled $\frac{1}{4}$. Each arrow must thought to be labelled with a (formal) variable x. From each state there are exactly 4 arrows starting, according to the four possible inputs $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, 1. Observe that the loops have not been drawn in the figure.

However, for the computation, it is easier to neglect the loops at each state for a moment. In terms of generating functions that means that a function $f(x)$ (without the loops) turns into $\frac{1}{1-x}f(\frac{x}{1-x})$ (with the loops).

The *transposed* of the *transition matrix* is:

$$\begin{array}{c|cccccccc}
 & q_0 & \beta & p_0 & q_1 & p_1 & q_2 & p_2 & \cdots \\
\hline
q_0 & & x & x & x & & & & \\
\beta & x & & & & & & & \\
p_0 & x & x & & & x & & & \\
q_1 & x & x & x & & x & x & & \\
p_1 & & & x & x & & & x & \\
q_2 & & & & x & & & x & \ddots \\
p_2 & & & & & x & x & & \\
\vdots & & & & & & \ddots & & \ddots
\end{array}$$

or

$$\begin{pmatrix}
B_0 & C_0 & & & & \\
D_0 & B & C & & & \\
 & D & B & C & & \\
 & & D & B & C & \\
 & & & \ddots & \ddots & \ddots
\end{pmatrix}$$

with

$$B = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}, \; C = D = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix},$$

and

$$B_0 = \begin{pmatrix} 0 & x & x \\ x & 0 & 0 \\ x & x & 0 \end{pmatrix}, \; C_0 = \begin{pmatrix} x & 0 \\ 0 & 0 \\ 0 & x \end{pmatrix}, \; D_0 = \begin{pmatrix} x & x & x \\ 0 & 0 & x \end{pmatrix}.$$

In [3] there is a method to get the generating functions in such a situation automatically:

Consider first

$$Z_1 = C\,Z_1\,D\,Z_1 + B\,Z_1 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, by symmetry, we may write

$$Z_1 = \begin{pmatrix} f & g \\ g & f \end{pmatrix}$$

and so

$$\begin{pmatrix} f & g \\ g & f \end{pmatrix} = C \begin{pmatrix} f & g \\ g & f \end{pmatrix} D \begin{pmatrix} f & g \\ g & f \end{pmatrix} + B \begin{pmatrix} f & g \\ g & f \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From this we get the following two equations for $f(x)$ and $g(x)$:

$$\begin{aligned} f &= x^2(f^2+g^2) + xg + 1 \\ g &= 2x^2 fg + xf \end{aligned}$$

For simplicity we substitute as follows:

$$F = f + g,\ G = f - g,\ f = \frac{F+G}{2},\ g = \frac{F-G}{2}$$

The two equations turn now into

$$\begin{aligned} F &= x^2F^2 + xF + 1 \\ G &= x^2G^2 - xG + 1 \end{aligned}$$

The generating function

$$M(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}$$

is well-known from *Motzkin-statistics.* With this notation we may write

$$F(x) = M(x),\ G(x) = M(-x)$$

$$\begin{aligned} f &= \frac{1}{2}\Big(M(x) + M(-x)\Big) \\ g &= \frac{1}{2}\Big(M(x) - M(-x)\Big). \end{aligned}$$

The generating functions discussed so far refer to a transition graph which "forgets" about the irregular beginning. To handle this, we proceed as follows: We introduce generating functions, all of them describing path's starting in the state q_0 and landing in a specified state. Symbolically:

$$\begin{aligned} a_1 &\quad : \ q_0 \to q_0 \\ a_2 &\quad : \ q_0 \to \beta \\ a_3 &\quad : \ q_0 \to p_0 \\ b_i &\quad : \ q_0 \to q_{i+1} \\ c_i &\quad : \ q_0 \to p_{i+1} \end{aligned}$$

Also, let

$$S(0) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \text{and} \quad T(i) = \begin{pmatrix} b_i \\ c_i \end{pmatrix} \ \text{ for } \ i \ge 0.$$

Then we have the following equations for the generating functions of interest:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = S(0) = x^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f & g \\ g & f \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + x \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$T(i) = \begin{pmatrix} b_i \\ c_i \end{pmatrix} = x^i \begin{pmatrix} f & g \\ g & f \end{pmatrix}^i \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad (i \geq 0)$$

An easy proof by induction shows that

$$\begin{pmatrix} f & g \\ g & f \end{pmatrix}^i = \frac{1}{2} \begin{pmatrix} M(x)^i + M(-x)^i & M(x)^i - M(-x)^i \\ M(x)^i - M(-x)^i & M(x)^i + M(-x)^i \end{pmatrix}.$$

The function $\varphi(x)$, representing the *average empty space* may be expressed by these functions as follows:

$$4\,\varphi(x) = 3\,a_2 + 2\,a_3 + \sum_{i\geq 0}(i+1)\,b_i + \sum_{i\geq 0}(i+3)\,c_i.$$

Let us write the equations for a_1, a_2, a_3 once again:

$$\begin{aligned} a_1 &= x^2\big[f\,a_1 + f\,a_2 + f\,a_3 + g\,a_3\big] + x(a_2 + a_3) + 1 \\ a_2 &= x\,a_1 \\ a_3 &= x^2\big[g\,a_1 + g\,a_2 + g\,a_3 + f\,a_3\big] + x(a_1 + a_2) \end{aligned}$$

From now on we restrict ourselves to local expansions around $x \sim \frac{1}{3}$. Actually, there is another one at $x = -\frac{1}{3}$. However we must, as mentioned earlier, perform $f(x) \to \frac{1}{1-x} f(\frac{x}{1-x})$; then $\frac{1}{3}$ shifts to $\frac{1}{4}$, whereas $-\frac{1}{3}$ shifts away.

$$\begin{aligned} M(x) &\sim 3 - 3\sqrt{3}\sqrt{1-3x} \\ M(-x) &\sim 6 - 3\sqrt{3} \\ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} &\sim \frac{1}{\sqrt{1-3x}} \begin{pmatrix} \frac{3}{4}(\sqrt{3}-1) \\ \frac{\sqrt{3}-1}{4} \\ \frac{1}{2} \end{pmatrix} \\ \begin{pmatrix} b_0 \\ c_0 \end{pmatrix} &\sim \frac{1}{\sqrt{1-3x}} \begin{pmatrix} -\frac{5}{2} + 2\sqrt{3} \\ \frac{5}{2} - \sqrt{3} \end{pmatrix} \\ T(i) &\sim \frac{1}{2\sqrt{1-3x}} \begin{pmatrix} \sqrt{3} + (3\sqrt{3}-5)(2-\sqrt{3})^{i-1} \\ \sqrt{3} - (3\sqrt{3}-5)2-\sqrt{3})^{i-1} \end{pmatrix} \quad (i \geq 1) \end{aligned}$$

Now we come to our function $\varphi(x)$.

$$4\,\phi(x) \sim \sum_{i\geq 1} i\,b_i + \sum_{i\geq 1} i\,c_i$$

Let us recall an elementary formula:

$$\sum_{i\geq 1} i\,x^i = \frac{x}{(1-x)^2}$$

$$T(i) = \begin{pmatrix} b_i \\ c_i \end{pmatrix} = x^i \begin{pmatrix} f & g \\ g & f \end{pmatrix}^i \begin{pmatrix} a_1 + a_2 + a_3 \\ a_3 \end{pmatrix}$$

Therefore we find

$$\begin{aligned}
\sum_{i\ge 1} i\,b_i &= \frac{a_1+a_2+2a_3}{2}\sum_{i\ge 1} i(xF)^i + \frac{a_1+a_2}{2}\sum_{i\ge 1} i(xG)^i \\
&= \frac{a_1+a_2+2a_3}{2}\frac{xF}{(1-xF)^2} + \frac{a_1+a_2}{2}\frac{xG}{(1-xG)^2} \\
\sum_{i\ge 1} i\,c_i &= \frac{a_1+a_2+2a_3}{2}\sum_{i\ge 1} i(xF)^i - \frac{a_1+a_2}{2}\sum_{i\ge 1} i(xG)^i \\
&= \frac{a_1+a_2+2a_3}{2}\frac{xF}{(1-xF)^2} - \frac{a_1+a_2}{2}\frac{xG}{(1-xG)^2}.
\end{aligned}$$

Henceforth

$$\varphi \sim \frac{\sqrt{3}}{12}\big(1-3x\big)^{-3/2}.$$

Now we perform $f(x) \to \frac{1}{1-x} f(\frac{x}{1-x})$, changing the singularity of interest from $\frac13$ to $\frac14$. The transformed φ-function behaves around $\frac14$ like

$$\begin{aligned}
\varphi(x) &\sim \frac{\sqrt{3}}{12}\frac{1}{1-x}\Big(1-\frac{3x}{1-x}\Big)^{-3/2} \\
&\sim \frac{\sqrt{3}}{12}\frac{4}{3}\left(\frac{1-4x}{\frac34}\right)^{-3/2} \\
&\sim \frac{1}{8}\big(1-4x\big)^{-3/2}.
\end{aligned}$$

Let $[x^n]\varphi(x)$ denote, as usual, the coefficient of x^n in $\varphi(x)$. By *Darboux's technique* we may write

$$\begin{aligned}
E_n &\sim 4^{-n}\,\frac{1}{8}\,[x^n](1-4x)^{-3/2} \\
&\sim 4^{-n}\,\frac{1}{8}\,\frac{4^n\,n^{1/2}}{\Gamma(\frac32)} \\
&= \frac{1}{8}\frac{\sqrt{n}}{\frac12\sqrt{\pi}} \\
&= \frac{1}{4}\sqrt{\frac{n}{\pi}}.
\end{aligned}$$

This completes the proof.

References

1. P.Flajolet and J.Vitter, *Average-Case Analysis of Algorithms and Data Structures*, in: Handbook of Theoretical Computer Science, North-Holland, to appear 1990.
2. W. Knödel, *Über das mittlere Verhalten von on-line Packungsalgorithmen*, EIK **19** (1983), 427–433.
3. W.Kuich and F.J.Urbanek, *Infinite Linear Systems and One Counter Languages*, Theor.Comput. Sci. (1983), 95–126.

4. H.Prodinger, *Einige Bemerkungen zu einer Arbeit von W.Knödel über das mittlere Verhalten von on–line–Packungsalgorithmen*, EIK **21** (1985), 3–7.

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