

LOGARITHMS OF A BINOMIAL SERIES: A STIRLING NUMBER APPROACH

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ABSTRACT. The p -th power of the logarithm of the Catalan generating function is computed using the Stirling cycle numbers. Instead of Stirling numbers, one may write this generating function in terms of higher order harmonic numbers.

1. INTRODUCTION

Knuth [3] proposed the exciting formula

$$(\log C(z))^2 = \sum_{n \geq 1} \binom{2n}{n} (H_{2n-1} - H_n) \frac{z^n}{n},$$

where

$$C(z) = \frac{1 - \sqrt{1 - 4z}}{2z} = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} z^n \quad \text{and} \quad H_n = \sum_{1 \leq k \leq n} \frac{1}{k}$$

with the generating function of Catalan numbers and harmonic numbers.

This formula was recently extended by Chu [5] to general exponents p . Note that Knuth talked about the exponent 1 in his christmas lecture from 2014 [4].

We present here a very simple approach to this question using Stirling cycle numbers; recall [2] that they transform falling powers into ordinary powers viz.

$$x^n = \sum_{0 \leq k \leq n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} x^k.$$

2. THE EXPANSION OF THE p -TH POWER

The substitution $z = \frac{u}{(1+u)^2}$ was presented in [1] and it is extremely useful when dealing with Catalan numbers and Catalan statistics. Then $C(z) = 1 + u$, and, by the Lagrange inversion formula [7],

$$u^m = \sum_{n \geq m} \frac{m}{n} \binom{2n}{n-m} z^n$$

2010 *Mathematics Subject Classification.* 05A15; 05A10.

Key words and phrases. Catalan numbers, logarithm, generating function, Stirling number.

for $m \geq 1$. For $m = 0$ the formula is still true when taking a limit. We now consider the bivariate generating function

$$F(z, \alpha) = \sum_{p \geq 0} \frac{\alpha^p}{p!} (\log C(z))^p = \exp(\alpha \log C(z)) = C^\alpha(z) = (1+u)^\alpha = \sum_{m \geq 0} \binom{\alpha}{m} u^m.$$

But

$$\binom{\alpha}{m} = \frac{1}{m!} \alpha^m = \frac{1}{m!} \sum_{0 \leq k \leq m} (-1)^{m-k} \begin{bmatrix} m \\ k \end{bmatrix} \alpha^k.$$

Therefore

$$F(z, \alpha) = \sum_{0 \leq k \leq m} \frac{1}{m!} (-1)^{m-k} \begin{bmatrix} m \\ k \end{bmatrix} \alpha^k \sum_{n \geq m} \frac{m}{n} \binom{2n}{n-m} z^n.$$

The desired formula follows from reading off coefficients of α^p :

$$(\log C(z))^p = p! [\alpha^p] F(z, \alpha) = \sum_{p \leq m \leq n} \frac{p!}{m!} (-1)^{m-p} \begin{bmatrix} m \\ p \end{bmatrix} \frac{m}{n} \binom{2n}{n-m} z^n.$$

3. SPECIAL CASES

For $p = 1$, we get the instance of the Christmas lecture:

$$\log C(z) = [\alpha^1] F(z, \alpha) = \sum_{1 \leq m \leq n} \frac{1}{m!} (-1)^{m-1} \begin{bmatrix} m \\ 1 \end{bmatrix} \frac{m}{n} \binom{2n}{n-m} z^n.$$

Since $\begin{bmatrix} m \\ 1 \end{bmatrix} = (m-1)!$, this leads to

$$\log C(z) = [\alpha^1] F(z, \alpha) = \frac{1}{2} \sum_{n \geq 1} \frac{1}{n} \binom{2n}{n} z^n.$$

Now we turn to the instance $p = 2$ from [3]. (Note that $\begin{bmatrix} m \\ 2 \end{bmatrix} = (m-1)! H_{m-1}$.)

$$\begin{aligned} 2[\alpha^2] F(z, \alpha) &= \sum_{2 \leq m \leq n} \frac{2}{m!} (-1)^m \begin{bmatrix} m \\ 2 \end{bmatrix} \frac{m}{n} \binom{2n}{n-m} z^n \\ &= 2 \sum_{2 \leq m \leq n} H_{m-1} (-1)^m \frac{1}{n} \binom{2n}{n-m} z^n \\ &= 2 \sum_{1 \leq j < m \leq n} \frac{1}{j} (-1)^m \frac{1}{n} \binom{2n}{n-m} z^n \\ &= 2 \sum_{1 \leq j < n} \frac{1}{j} (-1)^{j-1} \frac{1}{n} \binom{2n-1}{n-j-1} z^n. \end{aligned}$$

To obtain the form proposed by Knuth, we still need to prove that

$$\binom{2n}{n} (H_{2n-1} - H_n) = 2 \sum_{1 \leq j < n} \frac{(-1)^{j-1}}{j} \binom{2n-1}{n-j-1}.$$

Modern computer algebra systems readily simplify the difference of these two sides to 0, as expected.

4. CONNECTION WITH HARMONIC NUMBERS — THE GENERAL CASE

In [6], there is the general formula

$$\frac{1}{n!} \begin{bmatrix} n+1 \\ r+1 \end{bmatrix} = (-1)^r \sum_{\{r\}} \prod_{j=1}^l \frac{(-1)^{i_j}}{i_j!} \left(\frac{H_n^{(r_j)}}{r_j} \right)^{i_j}.$$

Here, the sum is over all partitions of r : $r = i_1 r_1 + \cdots + i_l r_l$, with parts $r_1 > \cdots > r_l \geq 1$ and positive integers i_1, \dots, i_l . As an example, the partitions of $r = 4$ are 4, 3+1, 2+2, 2+1+1, 1+1+1+1, written alternatively as $1 \cdot 4$, $1 \cdot 3 + 1 \cdot 1$, $2 \cdot 2$, $1 \cdot 2 + 2 \cdot 1$, $4 \cdot 1$.

There appear higher order harmonic numbers as well:

$$H_n^{(i)} = \sum_{1 \leq k \leq n} \frac{1}{k^i}.$$

Here are the first few instances:

$$\begin{aligned} \frac{1}{n!} \begin{bmatrix} n+1 \\ 2 \end{bmatrix} &= H_n, \\ \frac{1}{n!} \begin{bmatrix} n+1 \\ 3 \end{bmatrix} &= -\frac{1}{2} H_n^{(2)} + \frac{1}{2} H_n^2, \\ \frac{1}{n!} \begin{bmatrix} n+1 \\ 4 \end{bmatrix} &= \frac{1}{3} H_n^{(3)} - \frac{1}{2} H_n^{(2)} H_n + \frac{1}{6} H_n^3, \\ \frac{1}{n!} \begin{bmatrix} n+1 \\ 5 \end{bmatrix} &= -\frac{1}{4} H_n^{(4)} + \frac{1}{3} H_n^{(3)} H_n + \frac{1}{8} (H_n^{(2)})^2 - \frac{1}{4} H_n^{(2)} H_n^2 + \frac{1}{24} H_n^4. \end{aligned}$$

This allows to replace $\frac{1}{(m-1)!} \begin{bmatrix} m \\ p \end{bmatrix}$ in

$$(\log C(z))^p = \sum_{p \leq m \leq n} \frac{1}{(m-1)!} \begin{bmatrix} m \\ p \end{bmatrix} (-1)^{m-p} \frac{p!}{n} \binom{2n}{n-m} z^n$$

by an expression involving $H_{m-1}^{(1)}, \dots, H_{m-1}^{(p-1)}$.

5. EXTENSION

If instead of $u = z(1+u)^2$ we work with $u = z(1+u)^\lambda$, then we deal with the generating function of extended (generalized) Catalan numbers

$$C_\lambda(z) = \sum_{n \geq 0} \binom{1+n\lambda}{n} \frac{z^n}{1+n\lambda}.$$

From [2], we infer that

$$u^m = \sum_{n \geq m} \binom{\lambda n + m}{n} \frac{m}{\lambda n + m} z^n.$$

So

$$\begin{aligned} F(z, \alpha) &= \sum_{p \geq 0} \frac{\alpha^p}{p!} (\log C_\lambda(z))^p = \exp(\alpha \log C_\lambda(z)) = C_\lambda^\alpha(z) = (1 + u)^\alpha = \sum_{m \geq 0} \binom{\alpha}{m} u^m \\ &= \sum_{0 \leq k \leq m \leq n} \frac{1}{m!} (-1)^{m-k} \begin{bmatrix} m \\ k \end{bmatrix} \alpha^k \binom{\lambda n + m}{n} \frac{m}{\lambda n + m} z^n. \end{aligned}$$

The desired formula follows from reading off coefficients of α^p :

$$(\log C_\lambda(z))^p = p! [\alpha^p] F(z, \alpha) = \sum_{p \leq m \leq n} \frac{p!}{m!} (-1)^{m-p} \begin{bmatrix} m \\ p \end{bmatrix} \binom{\lambda n + m}{n} \frac{m}{\lambda n + m} z^n.$$

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