Asymmetric generalizations of the Filbert matrix and variants

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Abstract. Four generalizations of the *Filbert matrix* are considered, with additional asymmetric parameter settings. Explicit formulæ are derived for the LU-decompositions, their inverses, and the inverse matrix. The approach is mainly to use q-analysis and to leave the justification of the necessary identities to the q-version of Zeilberger's algorithm for some of them, and for the rest of the necessary identities, to guess the relevant quantities and proving them later by induction.

1. Introduction

The Filbert matrix $\mathcal{F}_N = (\check{h}_{ij})_{i,j=1}^N$ is defined by $\check{h}_{ij} = \frac{1}{F_{i+j-1}}$ as an analogue of the Hilbert matrix where F_n is the nth Fibonacci number. It has been defined and studied by Richardson [**10**].

The Filbert matrix has been extended by Berg [**1**] and Ismail [**2**].

The present authors have generalized and extended this concept in a series of papers [**3, 4, 5, 6, 7, 9**] to matrices with entries

$$\frac{1}{F_{\lambda(i+j)+r}F_{\lambda(i+j+1)+r}\dots F_{\lambda(i+j+k-1)+r}} \quad \text{and} \quad \frac{1}{L_{\lambda(i+j)+r}\dots L_{\lambda(i+j+k-1)+r}}.$$

Here, $\lambda, k \geqslant 1$ and $r \geqslant -1$ are integer parameters and L_n are Lucas numbers.

In another direction [**5**], the matrices with entries

$$g_{ij} = \frac{F_{\lambda(i+j)+r}}{F_{\lambda(i+j)+s}} \quad \text{and} \quad v_{ij} = \frac{L_{\lambda(i+j)+r}}{L_{\lambda(i+j)+s}}$$

were introduced; here s, r and λ are integer parameters such that $s \neq r$, and $r, s \geqslant -1$ and $\lambda \geqslant 1$. This was the first nontrivial instance where the numerator of the entries is *not* equal to one.

All these extensions were driven by the search for "nice" explicit results: explicit formulæ for the LU-decomposition, their inverses, and the Cholesky factorization.

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Here, we go one step further, by allowing an *asymmetric growth of indices.* We, however, confine ourselves to $k=1$; for this instance, the inverse matrix also enjoys nice closed form entries, which is no longer true for $k \geqslant 2$. To be more specific, we introduce four generalizations of the Filbert matrix $\mathcal{F}$, and define the matrices $\mathcal{T}$, $\mathcal{M}$, $\mathcal{H}$ and $\mathcal{W}$ with entries

$$t_{ij} = \frac{1}{F_{\lambda i+\mu j+r}},\ m_{ij} = \frac{F_{\lambda i+\mu j+r}}{F_{\lambda i+\mu j+s}},\ h_{ij} = \frac{1}{L_{\lambda i+\mu j+r}} \text{ and } w_{ij} = \frac{L_{\lambda i+\mu j+r}}{L_{\lambda i+\mu j+s}},$$

respectively, where s, r, λ and μ are integer parameters such that $s \neq r$, and $r, s \geqslant -1$ and $\lambda, \mu \geqslant 1$.

Of course, because of these asymmetric entries, we cannot get a Cholesky decomposition anymore.

Now we discuss our settings. Let $\{U_n\}$ and $\{V_n\}$ be generalized Fibonacci and Lucas sequences, respectively, whose Binet forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n(1+q^n)$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$.

When $\alpha = \frac{1+\sqrt{5}}{2}$ (or equivalently $q = (1-\sqrt{5})/(1+\sqrt{5})$), the sequence $\{U_n\}$ is reduced to the Fibonacci sequence $\{F_n\}$ and the sequence $\{V_n\}$ is reduced to the Lucas sequence $\{L_n\}$.

When $\alpha = 1+\sqrt{2}$ (or equivalently $q = (1-\sqrt{2})/(1+\sqrt{2})$), the sequence $\{U_n\}$ is reduced to the Pell sequence $\{P_n\}$ and the sequence $\{V_n\}$ is reduced to the Pell-Lucas sequence $\{Q_n\}$.

We will mostly deal with the q-forms; translating the results back to the the Fibonacci and Lucas world, say, is easy: We only have to systematically replace $1-q^n$ by $\frac{1-q}{\alpha^{n-1}}F_n$ and $1+q^n$ by $\alpha^n L_n$ and replace what is eventually left by its numerical values.

Throughout this paper we will use the notation of the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$.

We rewrite the entries of the matrices $\mathcal{T}$, $\mathcal{M}$, $\mathcal{H}$ and $\mathcal{W}$ in q-form:

$$\begin{aligned}
t_{ij} &= \mathbf{i}^{1-r-\lambda i-\mu j} q^{\frac{1}{2}(\lambda i+\mu j+r-1)} \frac{1-q}{1-q^{\lambda i+\mu j+r}},\\
m_{ij} &= \mathbf{i}^{r-s} q^{\frac{s-r}{2}} \frac{1-q^{\lambda i+\mu j+r}}{1-q^{\lambda i+\mu j+s}},\\
h_{ij} &= \mathbf{i}^{-\lambda i-\mu j-r} q^{-\frac{1}{2}(\lambda i+\mu j+r)} \frac{1}{1+q^{\lambda i+\mu j+r}},\\
w_{ij} &= \mathbf{i}^{r-s} q^{\frac{s-r}{2}} \frac{1+q^{\lambda i+\mu j+r}}{1+q^{\lambda i+\mu j+s}}.
\end{aligned}$$

In each of the four instances, we consider the LU-decomposition of the matrix $M = LU$. We are able to get explicit formulæ for L, U, L^{-1}, U^{-1} and M^{-1}.

As it was noted in the above cited earlier papers, the sizes of the matrices do not really matter, and they can be thought as infinite matrices and we may restrict them whenever necessary to the first N rows resp. columns and write $\mathcal{T}_N$ etc. This is not true for the inverse matrices; here, the entries *depend* on N.

All our identities hold for general q, and results about Fibonacci and Lucas numbers come out as corollaries for the special choice of q, as explained.

The important part is to *find* the explicit forms. This was done by experiments with a computer algebra system and spotting patterns. This becomes increasingly complicated when more and more new parameters are introduced, as the guessing only works for fixed choices of the parameters, and one needs to vary them as well.

Once one *knows* how the entries look like, proofs are by reducing sums to single terms. For this, the q-Zeilberger algorithm is a handy tool. However for the matrices $\mathcal{M}$ and $\mathcal{W}$, the present versions of the q-Zeilberger algorithm do not work, and we have to *simulate* it by noticing that the relevant sums are Gosper-summable. To do this, some more guessing (with an additional parameter) is required. Consequently, since all these proofs are routine and somewhat tedious, we only present two typical examples. It would be a good student project to work them all out in full detail.

As an illustration, we always write out the Fibonacci/Lucas cases for $\lambda, \mu \in \{2, 3\}$ and $r = 1$, $s = -1$.

2. The matrix $\mathcal{T}$

We obtain the LU-decomposition $\mathcal{T} = L \cdot U$:

Theorem 2.1. *For $1 \leqslant d \leqslant n$ we have*

$$L_{n,d} = \mathbf{i}^{\lambda(d-n)} q^{-\frac{1}{2}\lambda(d-n)} \frac{(q^\lambda; q^\lambda)_{n-1} (q^{r+\mu+\lambda d}; q^\mu)_d}{(q^\lambda; q^\lambda)_{n-d} (q^\lambda; q^\lambda)_{d-1} (q^{r+\mu+\lambda n}; q^\mu)_d}.$$

Fibonacci Corollary for $\lambda = 3$, $\mu = 2$ and $r = 1$:

Corollary 2.1. *For $1 \leqslant d \leqslant n$,*

$$L_{n,d} = \frac{\Big(\prod_{t=1}^{n-1} F_{3t}\Big)\Big(\prod_{t=1}^{d} F_{2t+3d+1}\Big)}{\Big(\prod_{t=1}^{n-d} F_{3t}\Big)\Big(\prod_{t=1}^{d-1} F_{3t}\Big)\Big(\prod_{t=1}^{d} F_{2t+3d+1}\Big)}.$$

Theorem 2.2. *For $1 \leqslant d \leqslant n$ we have*

$$\begin{aligned} U_{d,n} &= \mathbf{i}^{1-r-\lambda d-\mu n} q^{\mu \frac{n-d}{2} + rd - \frac{r}{2} + (\lambda+\mu)\frac{d^2}{2} - \frac{1}{2}} \\ &\quad \times \frac{(q^\lambda; q^\lambda)_{d-1} (1-q)}{(q^{r+\mu n+\lambda}; q^\lambda)_d (q^{r+\mu+\lambda d}; q^\lambda)_{d-1}} \frac{(q^\mu; q^\mu)_{n-1}}{(q^\mu; q^\mu)_{n-d}}. \end{aligned}$$

Fibonacci Corollary for $\lambda = 3$, $\mu = 2$ and $r = 1$:

Corollary 2.2. *For* $1 \leqslant d \leqslant n$

$$U_{d,n}=\frac{(-1)^{\binom{d}{2}+d-1}\Big(\prod_{t=1}^{d-1}F_{3t}\Big)\Big(\prod_{t=1}^{n-1}F_{2t}\Big)}{\Big(\prod_{t=1}^{d}F_{3t+2n+1}\Big)\Big(\prod_{t=1}^{n-d}F_{2t}\Big)\Big(\prod_{t=1}^{d-1}F_{2t+3d+1}\Big)}.$$

The inverses of the matrices L and U:

Theorem 2.3. *For* $1 \leqslant d \leqslant n$ *we have*

$$L^{-1}_{n,d}=(-1)^{n-d}\mathbf{i}^{\lambda(d-n)}q^{\lambda\frac{(n-d)}{2}^2}\frac{(q^\lambda;q^\lambda)_{n-1}}{(q^\lambda;q^\lambda)_{n-d}(q^\lambda;q^\lambda)_{d-1}}\frac{(q^{r+\mu+\lambda d};q^\mu)_{n-1}}{(q^{r+\mu+\lambda n};q^\mu)_{n-1}}.$$

Fibonacci Corollary for $\lambda=3$, $\mu=2$ and $r=1$:

Corollary 2.3. *For* $1 \leqslant d \leqslant n$

$$L^{-1}_{n,d}=\frac{(-1)^{\binom{n-d+1}{2}}\Big(\prod_{t=1}^{n-1}F_{3t}\Big)\Big(\prod_{t=1}^{n-1}F_{2t+3j+1}\Big)}{\Big(\prod_{t=1}^{n-d}F_{3t}\Big)\Big(\prod_{t=1}^{d-1}F_{3t}\Big)\Big(\prod_{t=1}^{n-1}F_{2t+3n+1}\Big)}.$$

Theorem 2.4. *For* $1 \leqslant d \leqslant n$ *we have*

$$\begin{aligned}U^{-1}_{d,n}&=q^{-\lambda\frac{n^2}{2}+\mu\frac{d^2}{2}-\mu nd-rn+\frac{r}{2}+\frac12}(-1)^{n+d}\mathbf{i}^{r-1+\lambda n+\mu d}\\&\times\frac{(q^{r+\lambda+\mu d};q^\lambda)_{n-1}(q^{r+\mu+\lambda n};q^\mu)_n}{(1-q)(q^\lambda;q^\lambda)_{n-1}(q^\mu;q^\mu)_{n-d}(q^\mu;q^\mu)_{d-1}}.\end{aligned}$$

Its Fibonacci Corollary for $\lambda=3$, $\mu=2$ and $r=1$:

Corollary 2.4. *For* $1 \leqslant d \leqslant n$

$$U^{-1}_{d,n}=(-1)^{\binom{n}{2}+d-1}\frac{\Big(\prod_{t=1}^{n-1}F_{3t+2d+1}\Big)\Big(\prod_{t=1}^{n}F_{2t+3n+1}\Big)}{\Big(\prod_{t=1}^{n-1}F_{3t}\Big)\Big(\prod_{t=1}^{n-d}F_{2t}\Big)\Big(\prod_{t=1}^{d-1}F_{2t}\Big)}.$$

As a consequence, we can compute the determinant of $\mathcal{T}_N$, since it is simply evaluated as $U_{1,1}\cdots U_{N,N}$:

Theorem 2.5.

$$\begin{aligned}\det\mathcal{T}_N&=\mathbf{i}^{(\mu-\lambda)\binom{N+1}{2}+N(1-r)}q^{\frac12N^2(\mu+r)-\frac12N+\frac16\mu N(N^2-1)+\frac1{12}\lambda N(N+1)(2N+1)}\\&\times(1-q)^N\prod_{d=1}^{N}\frac{(q^\lambda;q^\lambda)_{d-1}(q^\mu;q^\mu)_{d-1}}{(q^{r+\lambda+\mu d};q^\lambda)_d(q^{r+\mu+\lambda d};q^\mu)_{d-1}}.\end{aligned}$$

Fibonacci Corollary for $\lambda=3$, $\mu=2$ and $r=1$:

Corollary 2.5.

$$\det \mathcal{T}_N = (-1)^{\binom{N+2}{3}+N} \prod_{d=1}^{N} \frac{1}{F_{3d+2N+1}} \prod_{t=1}^{d-1} \frac{F_{3t}F_{2t}}{F_{3t+2N+1}F_{2t+3d+1}}.$$

Now we compute the inverse of the matrix $\mathcal{T}$. It depends on the dimension, so we compute $(\mathcal{T}_N)^{-1}$.

Theorem 2.6. *For $1 \leqslant n, d \leqslant N$:*

$$(\mathcal{T}_N)^{-1}_{n,d} = (-1)^{n-d}\mathbf{i}^{r-1+\mu n+\lambda d} q^{\mu\frac{n^2}{2}+\lambda\frac{d^2}{2}-\mu Nn-\lambda Nd-Ns+\frac{r}{2}+\frac{1}{2}}$$
$$\times \frac{(q^{r+\mu n+\lambda};q^\lambda)_N(q^{r+\lambda d+\mu};q^\mu)_N}{(q^\lambda;q^\lambda)_{N-d}(q^\lambda;q^\lambda)_{d-1}(q^\mu;q^\mu)_{N-n}(q^\mu;q^\mu)_{n-1}} \frac{1}{1-q^{r+\mu n+\lambda d}} \frac{1}{1-q}.$$

Remark. The inverse matrix and the other inverse matrices presented in this paper were *not* computed using the inverses of L and U, but rather obtained directly by our usual guessing strategy. While the first alternative would mean that we would have to simplify a sum, the second approach stays within our chosen method.

Fibonacci Corollary for $\lambda = 3$, $\mu = 2$ and $r = 1$:

Corollary 2.6. *For $1 \leqslant n, d \leqslant N$:*

$$(\mathcal{T}_N)^{-1}_{n,d} = \frac{(-1)^{\binom{n+1}{2}+(N+1)(n+1)+d}\Big(\prod_{t=1}^{N} F_{2t+3d+1}\Big)\Big(\prod_{t=1}^{N} F_{3t+2n+1}\Big)}{\Big(\prod_{t=1}^{N-n} F_{2t}\Big)\Big(\prod_{t=1}^{n-1} F_{2t}\Big)\Big(\prod_{t=1}^{N-d} F_{3t}\Big)\Big(\prod_{t=1}^{d-1} F_{3t}\Big)} \frac{1}{F_{2n+3d+1}}.$$

3. The matrix $\mathcal{M}$

We obtain the LU-decomposition $\mathcal{M} = L \cdot U$:

Theorem 3.1. *For $1 \leqslant d \leqslant n$ we have*

$$L_{n,d} = \frac{(q^\lambda;q^\lambda)_{n-1}}{(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{n-d}} \frac{(q^{s+\lambda d+\mu};q^\mu)_d}{(q^{s+\lambda n+\mu};q^\mu)_d} \frac{1-q^{(\lambda+\mu)\frac{d(d+1)}{2}+\lambda(n-d)+r+s(d-1)}}{1-q^{(\lambda+\mu)\frac{d(d+1)}{2}+r+s(d-1)}}.$$

Fibonacci Corollary for $\lambda = 2$, $\mu = 3$, $r = 1$ and $s = -1$:

Corollary 3.1. *For $1 \leqslant d \leqslant n$,*

$$L_{n,d} = \frac{\Big(\prod_{t=1}^{d-1} F_{2t}\Big)\Big(\prod_{t=1}^{d} F_{3t+2d}\Big)}{\Big(\prod_{t=1}^{d-1} F_{2t}\Big)\Big(\prod_{t=1}^{n-d} F_{2t}\Big)\Big(\prod_{t=1}^{d} F_{3t+2n-1}\Big)} \frac{F_{5d(d+1)/2+2n-3d+2}}{F_{5d(d+1)/2-d+2}}.$$

Theorem 3.2. *For $1 \leqslant d \leqslant n$ we have*

$$U_{d,n} = \mathbf{i}^{r-s} q^{(\lambda+\mu)\frac{d(d-1)}{2} - \frac{r+s}{2} + ds} \frac{(q^\lambda; q^\lambda)_{d-1} (q^\mu; q^\mu)_{n-1}}{(q^\mu; q^\mu)_{n-d} (q^{s+\lambda+n\mu}; q^\lambda)_d (q^{s+\mu+\lambda d}; q^\mu)_{d-1}}$$
$$\times \frac{1 - q^{(\lambda+\mu)\frac{(d+1)d}{2} + \mu(n-d) + r + s(d-1)}}{1 - q^{(\lambda+\mu)\frac{d(d-1)}{2} - 2s + r + ds}} (1 - q^{r-s}).$$

Fibonacci Corollary for $\lambda = 2$, $\mu = 3$, $r = 1$ and $s = -1$:

Corollary 3.2. *For $1 \leqslant d \leqslant n$*

$$U_{d,n} = (-1)^{d+1} (-1)^{\binom{d}{2}} \frac{\left(\prod_{t=1}^{d-1} F_{2t}\right)\left(\prod_{t=1}^{n-1} F_{3t}\right)}{\left(\prod_{t=1}^{n-d} F_{3t}\right)\left(\prod_{t=1}^{d-1} F_{3t+2d-1}\right)\left(\prod_{t=1}^{d} F_{2t+3n-1}\right)}$$
$$\times \frac{F_{5d(d+1)/2+3n-4d+2}}{F_{5d(d-1)/2-d+3}}.$$

The inverses of the matrices L and U:

Theorem 3.3. *For $1 \leqslant d \leqslant n$ we have*

$$L_{n,d}^{-1} = (-1)^{n-d} q^{\lambda\frac{n(n-1)}{2} + \lambda\frac{d(d+1)}{2} - \lambda nd} \frac{1 - q^{-\lambda d + \frac{\lambda+\mu}{2} n^2 + \frac{\lambda-\mu}{2} n + r + s(n-2)}}{1 - q^{-\lambda n + \frac{\lambda+\mu}{2} n^2 + \frac{\lambda-\mu}{2} n + r + s(n-2)}}$$
$$\times \frac{(q^\lambda; q^\lambda)_{n-1}}{(q^\lambda; q^\lambda)_{d-1} (q^\lambda; q^\lambda)_{n-d}} \frac{(q^{s+\lambda d+\mu}; q^\mu)_{n-1}}{(q^{s+\lambda n+\mu}; q^\mu)_{n-1}}.$$

Fibonacci Corollary for $\lambda = 2$, $\mu = 3$, $r = 1$ and $s = -1$:

Corollary 3.3. *For $1 \leqslant d \leqslant n$*

$$L_{n,d}^{-1} = \frac{(-1)^{n-d}\left(\prod_{t=1}^{n-1} F_{2t}\right)\left(\prod_{t=1}^{n-1} F_{3t+2d-1}\right)}{\left(\prod_{t=1}^{d-1} F_{2t}\right)\left(\prod_{t=1}^{n-d} F_{2t}\right)\left(\prod_{t=1}^{n-1} F_{3t+2n-1}\right)} \frac{F_{5n(n+1)/2-4n-2d+3}}{F_{5n(n-1)/2-n+3}}.$$

Theorem 3.4. *For $1 \leqslant d \leqslant n$ we have*

$$U_{d,n}^{-1} = \mathbf{i}^{s-r} (-1)^{n-d} q^{-\lambda\frac{n(n-1)}{2} + \mu\frac{d(d+1)}{2} - \mu nd + \frac{r+s}{2} - ns} \frac{1}{(1 - q^{r-s})}$$
$$\times \frac{(q^{\lambda+\mu d+s}; q^\lambda)_{n-1} (q^{\mu+\lambda n+s}; q^\mu)_n}{(q^\lambda; q^\lambda)_{n-1} (q^\mu; q^\mu)_{n-d} (q^\mu; q^\mu)_{d-1}} \frac{1 - q^{(\lambda+\mu)\frac{n(n-1)}{2} + \mu(n-d) + r + s(n-2)}}{1 - q^{(\lambda+\mu)\frac{n(n+1)}{2} + r + s(n-1)}}.$$

Its Fibonacci Corollary for $\lambda = 2$, $\mu = 3$, $r = 1$ and $s = -1$:

Corollary 3.4. *For* $1 \leqslant d \leqslant n$

$$U_{d,n}^{-1} = (-1)^{\binom{d+2}{2}+nd} \frac{\left(\prod_{t=1}^{n-1} F_{2t+3d-1}\right)\left(\prod_{t=1}^{n} F_{3t+2n-1}\right)}{\left(\prod_{t=1}^{d-1} F_{3t}\right)\left(\prod_{t=1}^{n-1} F_{2t}\right)\left(\prod_{t=1}^{n-d} F_{3t}\right)} \times \frac{F_{5n(n-1)/2+2n-3d+3}}{F_{5n(n+1)/2-n+2}}.$$

As a consequence, we can compute the determinant of $\mathcal{M}_N$, since it is simply evaluated as $U_{1,1}\cdots U_{N,N}$:

Theorem 3.5.

$$\det \mathcal{M}_N = \mathbf{i}^{N(r-s)} q^{\frac{1}{6}(\lambda+\mu)N(N^2-1)+\frac{1}{3}N(Ns+s-2r)}(1-q^{r-s})^N \times \prod_{d=1}^{N} \frac{(q^\lambda;q^\lambda)_{d-1}(q^\mu;q^\mu)_{d-1}}{(q^{\lambda+d\mu+s};q^\lambda)_d(q^{\mu+\lambda d+s};q^\mu)_{d-1}} \frac{1-q^{(\lambda+\mu)\frac{(d+1)d}{2}+r+s(d-1)}}{1-q^{(\lambda+\mu)\frac{d(d-1)}{2}-2s+r+ds}}.$$

Fibonacci Corollary for $\lambda = 2$, $\mu = 3$, $r = 1$ and $s = -1$:

Corollary 3.5.

$$\det \mathcal{M}_N = (-1)^{N+\frac{1}{6}N(N^2-1)} \prod_{d=1}^{N} \frac{F_{5d(d+1)/2-d+2}}{F_{5d(d-1)/2-d+3}} \prod_{t=1}^{d-1} \frac{\left(\prod_{t=1}^{d-1} F_{2t}\right)\left(\prod_{t=1}^{d-1} F_{3t}\right)}{\left(\prod_{t=1}^{d-1} F_{3t+2d-1}\right)\left(\prod_{t=1}^{d} F_{2t+3d-1}\right)}.$$

Now we compute the inverse of the matrix $\mathcal{M}$. It depends on the dimension, so we compute $(\mathcal{M}_N)^{-1}$.

Theorem 3.6. *For* $1 \leqslant n, d \leqslant N$*:*

$$(\mathcal{M}_N)_{n,d}^{-1} = (-1)^{n-d}\mathbf{i}^{s-r} q^{\frac{r+s}{2}+\lambda\frac{d(d+1)}{2}+\mu\frac{n(n+1)}{2}-Ns-Nd\lambda-Nn\mu} \times \frac{(q^{s+\lambda+\mu n};q^\lambda)_N(q^{s+\mu+\lambda d};q^\mu)_N}{(q^\lambda;q^\lambda)_{N-d}(q^\lambda;q^\lambda)_{d-1}(q^\mu;q^\mu)_{n-1}(q^\mu;q^\mu)_{N-n}} \times \frac{1}{(1-q^{s+\mu n+\lambda d})(1-q^{r-s})} \frac{1-q^{(\lambda+\mu)\frac{N(N+1)}{2}-\lambda d-\mu n+r+sN-2s}}{1-q^{(\lambda+\mu)\frac{N(N+1)}{2}+r+sN-s}}.$$

Fibonacci Corollary for $\lambda = 2$, $\mu = 3$, $r = 1$ and $s = -1$:

Corollary 3.6. *For* $1 \leqslant n, d \leqslant N$*:*

$$(\mathcal{M}_N)_{n,d}^{-1} = \frac{(-1)^{\binom{n+1}{2}+d+n+Nn}\left(\prod_{t=1}^{N} F_{3t+2d-1}\right)\left(\prod_{t=1}^{N} F_{2t+3n-1}\right)}{\left(\prod_{t=1}^{d-1} F_{2t}\right)\left(\prod_{t=1}^{n-1} F_{3t}\right)\left(\prod_{t=1}^{N-d} F_{2t}\right)\left(\prod_{t=1}^{N-n} F_{3t}\right)F_{2d+3n-1}} \times \frac{F_{5N(N+1)/2-2d-3n-N+3}}{F_{5N(N+1)/2-N+2}}.$$

4. The matrix $\mathcal{H}$

We obtain the LU-decomposition $\mathcal{H} = L\cdot U$:

THEOREM 4.1. *For $1 \leqslant d \leqslant n$ we have*

$$L_{n,d} = q^{\lambda\frac{n-d}{2}}\mathbf{i}^{-\lambda(n-d)}\frac{(-q^{r+\mu+\lambda d};q^{\mu})_d}{(-q^{r+\mu+\lambda n};q^{\mu})_d}\frac{(q^{\lambda};q^{\lambda})_{n-1}}{(q^{\lambda};q^{\lambda})_{d-1}(q^{\lambda};q^{\lambda})_{n-d}}.$$

Fibonacci Corollary for $\lambda=3$, $\mu=2$ and $r=1$:

COROLLARY 4.1. *For $1 \leqslant d \leqslant n$,*

$$L_{n,d} = \frac{\Big(\prod\limits_{t=1}^{d} L_{2t+3d+1}\Big)}{\Big(\prod\limits_{t=1}^{d} L_{2t+3n+1}\Big)}\frac{\Big(\prod\limits_{t=1}^{n-1} F_{3t}\Big)}{\Big(\prod\limits_{t=1}^{d-1} F_{3t}\Big)\Big(\prod\limits_{t=1}^{n-d} F_{3t}\Big)}.$$

THEOREM 4.2. *For $1 \leqslant d \leqslant n$ we have*

$$U_{d,n} = (-1)^{d-1}\mathbf{i}^{-r-\mu n-\lambda d}q^{\mu\frac{n-d}{2}+rd-\frac{r}{2}+(\lambda+\mu)\frac{d^2}{2}} \times \frac{(q^{\lambda};q^{\lambda})_{d-1}}{(-q^{r+\mu n+\lambda};q^{\lambda})_d(-q^{r+\mu+\lambda d};q^{\mu})_{d-1}}\frac{(q^{\mu};q^{\mu})_{n-1}}{(q^{\mu};q^{\mu})_{n-d}}.$$

Fibonacci Corollary for $\lambda=3$, $\mu=2$ and $r=1$:

COROLLARY 4.2. *For $1 \leqslant d \leqslant n$*

$$U_{d,n} = \frac{(-1)^{\binom{d}{2}}5^{d-1}}{\Big(\prod\limits_{t=1}^{d} L_{3t+2n+1}\Big)\Big(\prod\limits_{t=1}^{d-1} L_{2t+3d+1}\Big)}\frac{\Big(\prod\limits_{t=1}^{d-1} F_{3t}\Big)\Big(\prod\limits_{t=1}^{n-1} F_{2t}\Big)}{\Big(\prod\limits_{t=1}^{n-d} F_{2t}\Big)}.$$

The inverses of the matrices L and U:

THEOREM 4.3. *For $1 \leqslant d \leqslant n$ we have*

$$L^{-1}_{n,d} = (-1)^{n-d}\mathbf{i}^{-\lambda(n-d)}q^{\lambda\frac{(n-d)^2}{2}}\frac{(-q^{r+\mu+\lambda d};q^{\mu})_{n-1}}{(-q^{r+\mu+\lambda n};q^{\mu})_{n-1}}\frac{(q^{\lambda};q^{\lambda})_{n-1}}{(q^{\lambda};q^{\lambda})_{d-1}(q^{\lambda};q^{\lambda})_{n-d}}.$$

Fibonacci Corollary for $\lambda=3$, $\mu=2$ and $r=1$:

COROLLARY 4.3. *For $1 \leqslant d \leqslant n$*

$$L^{-1}_{n,d} = (-1)^{\binom{n+1}{2}+\binom{d}{2}-nd}\frac{\Big(\prod\limits_{t=1}^{n-1} L_{2t+3d+1}\Big)\Big(\prod\limits_{t=1}^{n-1} F_{3t}\Big)}{\Big(\prod\limits_{t=1}^{n-1} L_{2t+3n+1}\Big)\Big(\prod\limits_{t=1}^{d-1} F_{3t}\Big)\Big(\prod\limits_{t=1}^{n-d} F_{3t}\Big)}.$$

THEOREM 4.4. *For $1 \leqslant d \leqslant n$ we have*

$$U^{-1}_{d,n} = (-1)^{d-1}\mathbf{i}^{r+\lambda n+\mu d}q^{-\lambda\frac{n^2}{2}+\mu\frac{d^2}{2}-\mu nd-rn+\frac{r}{2}} \times \frac{(-q^{r+\mu d+\lambda};q^{\lambda})_{n-1}(-q^{r+\lambda n+\mu};q^{\mu})_n}{(q^{\lambda};q^{\lambda})_{n-1}(q^{\mu};q^{\mu})_{d-1}(q^{\mu};q^{\mu})_{n-d}}.$$

Its Fibonacci Corollary for $\lambda = 3$, $\mu = 2$ and $r = 1$:

Corollary 4.4. *For* $1 \leqslant d \leqslant n$

$$U_{d,n}^{-1} = (-1)^{\binom{n+1}{2}-d} \frac{\Big(\prod_{t=1}^{n} L_{2t+3n+1}\Big)\Big(\prod_{t=1}^{n-1} L_{3t+2d+1}\Big)}{5^{n-1}\Big(\prod_{t=1}^{d-1} F_{2t}\Big)\Big(\prod_{t=1}^{n-d} F_{2t}\Big)\Big(\prod_{t=1}^{n-1} F_{3t}\Big)}.$$

As a consequence, we can compute the determinant of $\mathcal{H}_N$, since it is simply evaluated as $U_{1,1} \cdots U_{N,N}$:

Theorem 4.5.

$$\det \mathcal{H}_N = \mathbf{i}^{\frac{1}{2}N(N+1)(\lambda+\mu)+Nr+N(N+1)} q^{\frac{N(\lambda+\mu)}{12}+\frac{N^2(\lambda+\mu)}{4}+\frac{N^3(\lambda+\mu)}{6}+\frac{N^2 r}{2}} \times \prod_{d=1}^{N} \frac{(q^\lambda;q^\lambda)_{d-1}(q^\mu;q^\mu)_{d-1}}{(-q^{r+\mu d+\lambda};q^\lambda)_d(-q^{r+\mu+\lambda d};q^\mu)_{d-1}}.$$

Fibonacci Corollary for $\lambda = 3$, $\mu = 2$ and $r = 1$:

Corollary 4.5.

$$\det \mathcal{H}_N = 5^{N(N-1)/2}(-1)^{\binom{N+1}{2}} \prod_{d=1}^{N} \frac{\Big(\prod_{t=1}^{d-1} F_{3t}\Big)\Big(\prod_{t=1}^{d-1} F_{2t}\Big)}{\Big(\prod_{t=1}^{d} L_{3t+2d+1}\Big)\Big(\prod_{t=1}^{d-1} L_{2t+3d+1}\Big)}.$$

Now we compute the inverse of the matrix $\mathcal{H}$. It depends on the dimension, so we compute $(\mathcal{H}_N)^{-1}$.

Theorem 4.6. *For* $1 \leqslant n, d \leqslant N$*:*

$$(\mathcal{H}_N)_{n,d}^{-1} = (-1)^{N-1-n-d} \mathbf{i}^{r+\mu n+\lambda d} q^{\mu\frac{n^2}{2}+\lambda\frac{d^2}{2}-\mu Nn-\lambda Nd-Nr+\frac{r}{2}} \times \frac{(-q^{r+\mu n+\lambda};q^\lambda)_N(-q^{r+\lambda d+\mu};q^\mu)_N}{(q^\lambda;q^\lambda)_{N-d}(q^\lambda;q^\lambda)_{d-1}(q^\mu;q^\mu)_{N-n}(q^\mu;q^\mu)_{n-1}} \frac{1}{1+q^{r+\mu n+\lambda d}}.$$

Fibonacci Corollary for $\lambda = 3$, $\mu = 2$ and $r = 1$:

Corollary 4.6. *For* $1 \leqslant n, d \leqslant N$*:*

$$(\mathcal{H}_N)_{n,d}^{-1} = (-1)^{\binom{d+1}{2}+Nd+n+d} \frac{\Big(\prod_{t=1}^{N} L_{3t+2n+1}\Big)\Big(\prod_{t=1}^{N} L_{2t+3d+1}\Big)}{5^{N-1}\Big(\prod_{t=1}^{N-d} F_{3t}\Big)\Big(\prod_{t=1}^{d-1} F_{3t}\Big)\Big(\prod_{t=1}^{N-n} F_{2t}\Big)\Big(\prod_{t=1}^{n-1} F_{2t}\Big)}.$$

5. The matrix $\mathcal{W}$

Now we collect our results related to the matrix $\mathcal{W}$.

For convenience, we use the same letters L, U, but with a different meaning.

We obtain the LU-decomposition $\mathcal{W} = L \cdot U$:

THEOREM 5.1. *For $1 \leqslant d \leqslant n$ we have*

$$L_{n,d} = \frac{(q^\lambda;q^\lambda)_{n-1}}{(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{n-d}} \frac{(-q^{s+\lambda d+\mu};q^\mu)_d}{(-q^{s+\lambda n+\mu};q^\mu)_d}$$
$$\times \frac{1-(-1)^d q^{(\lambda+\mu)\frac{d(d+1)}{2}+\lambda(n-d)+r+s(d-1)}}{1-(-1)^d q^{(\lambda+\mu)\frac{d(d+1)}{2}+r+s(d-1)}}.$$

Fibonacci Corollary for $\lambda = 2$, $\mu = 3$, $r = 1$ and $s = -1$:

COROLLARY 5.1. *For $1 \leqslant d \leqslant n$,*

$$L_{n,d} = \frac{\left(\prod_{t=1}^{n-1} F_{2t}\right)}{\left(\prod_{t=1}^{d-1} F_{2t}\right)\left(\prod_{t=1}^{n-d} F_{2t}\right)} \frac{\prod_{t=1}^{d} L_{3t+2d-1}}{\prod_{t=1}^{d} L_{3t+2n-1}}$$
$$\times \begin{cases} \frac{L_{5d(d+1)/2+2n-3d+2}}{L_{5d(d+1)/2-d+2}} & \text{if } d \text{ is even,} \\ \frac{F_{5d(d+1)/2+2n-3d+2}}{F_{5d(d+1)/2-d+2}} & \text{if } d \text{ is odd.} \end{cases}$$

THEOREM 5.2. *For $1 \leqslant d \leqslant n$ we have*

$$U_{d,n} = \frac{\mathbf{i}^{s-r}(-1)^{d-1} q^{+(\lambda+\mu)\frac{d(d-1)}{2}-\frac{r+s}{2}+ds}(q^\lambda;q^\lambda)_{d-1}(q^\mu;q^\mu)_{n-1}}{(q^\mu;q^\mu)_{n-d}(-q^{\lambda+n\mu+s};q^\lambda)_d(-q^{\mu+\lambda d+s};q^\mu)_{d-1}}$$
$$\times \frac{1-(-1)^d q^{(\lambda+\mu)\frac{(d+1)d}{2}+\mu(n-d)+r+ds-s}}{1+(-1)^d q^{(\lambda+\mu)\frac{d(d-1)}{2}+r+ds-2s}}(1-q^{r-s}).$$

Fibonacci Corollary for $\lambda = 2$, $\mu = 3$, $r = 1$ and $s = -1$:

COROLLARY 5.2. *For $1 \leqslant d \leqslant n$*

$$U_{d,n} = (-1)^{\binom{d}{2}} \frac{\left(\prod_{t=1}^{d-1} F_{2t}\right)\left(\prod_{t=1}^{n-1} F_{3t}\right)}{\left(\prod_{t=1}^{n-d} F_{3t}\right)\left(\prod_{t=1}^{d-1} L_{3t+2d-1}\right)\left(\prod_{t=1}^{d} L_{2t+3n-1}\right)}$$
$$\times \begin{cases} \frac{5^{d-1} L_{5(d+1)d/2+3n-4d+2}}{F_{5d(d-1)/2-d+3}} & \text{if } d \text{ is odd,} \\ \frac{5^{d} F_{5(d+1)d/2+3n-4d+2}}{L_{5d(d-1)/2-d+3}} & \text{if } d \text{ is even.} \end{cases}$$

The inverses of the matrices L and U:

THEOREM 5.3. *For $1 \leqslant d \leqslant n$ we have*

$$L^{-1}_{n,d} = \frac{(-1)^{n-d} q^{\lambda\binom{n-d}{2}}(q^{\lambda(n-d+1)};q^\lambda)_{d-1}(-q^{s+\mu+\lambda d};q^\mu)_{n-1}}{(q^\lambda;q^\lambda)_{d-1}(-q^{s+\mu+\lambda n};q^\mu)_{n-1}}$$
$$\times \frac{1+(-1)^n q^{(\lambda+\mu)\frac{n(n+1)}{2}-\mu n-\lambda d+r-2s+ns}}{1+(-1)^n q^{(\lambda+\mu)\frac{n(n+1)}{2}-\mu n-\lambda n+r-2s+ns}}.$$

Fibonacci Corollary for $\lambda = 2$, $\mu = 3$, $r = 1$ and $s = -1$:

Corollary 5.3. *For $1 \leqslant d \leqslant n$*

$$L^{-1}_{n,d} = \frac{(-1)^{n-d}\Big(\prod_{t=1}^{n-1} F_{2t}\Big)\Big(\prod_{t=1}^{n-1} L_{3t+2d-1}\Big)}{\Big(\prod_{t=1}^{d-1} F_{2t}\Big)\Big(\prod_{t=1}^{n-d} F_{2t}\Big)\Big(\prod_{t=1}^{n-1} L_{3t+2n-1}\Big)} \times \begin{cases} \frac{F_{5n(n+1)/2-2d-4n+3}}{F_{5n(n+1)/2-6n+3}} & \text{if } n \text{ is odd,} \\ \frac{L_{5n(n+1)/2-2d-4n+3}}{L_{5n(n+1)/2-6n+3}} & \text{if } n \text{ is even.} \end{cases}$$

Theorem 5.4. *For $1 \leqslant d \leqslant n$ we have*

$$\begin{aligned} U^{-1}_{d,n} &= q^{\frac{r+s}{2} - \frac{r}{2} + \mu\frac{d(d+1)}{2} - \lambda\frac{n(n-1)}{2} - ns - \mu nd} \\ &\times \frac{\mathbf{i}^{s-r}(-1)^{d-1}(-q^{s+\lambda+\mu d}; q^\lambda)_{n-1}(-q^{s+\mu+\lambda n}; q^\mu)_n}{(1-q^{r-s})(q^\lambda; q^\lambda)_{n-1}(q^\mu; q^\mu)_{n-d}(q^\mu; q^\mu)_{d-1}} \\ &\times \frac{1+(-1)^n q^{(\lambda+\mu)(n(n-1)/2)+\mu(n-d)+r+s(n-2)}}{1-(-1)^n q^{(\lambda+\mu)\frac{n(n+1)}{2}+r+s(n-1)}}. \end{aligned}$$

Its Fibonacci Corollary for $\lambda = 2$, $\mu = 3$, $r = 1$ and $s = -1$:

Corollary 5.4. *For $1 \leqslant d \leqslant n$*

$$U^{-1}_{d,n} = (-1)^{\binom{d+1}{2}+d+n-nd} \frac{\Big(\prod_{t=1}^{n-1} L_{2t+3d-1}\Big)\Big(\prod_{t=1}^{n} L_{3t+2n-1}\Big)}{\Big(\prod_{t=1}^{d-1} F_{3t}\Big)\Big(\prod_{t=1}^{n-1} F_{2t}\Big)\Big(\prod_{t=1}^{n-d} F_{3t}\Big)} \times \begin{cases} \frac{L_{5n(n-1)/2+2n-3d+3}}{5^n F_{5n(n+1)/2-n+2}} & \text{if } n \text{ is even,} \\ \frac{F_{5n(n-1)/2+2n-3d+3}}{5^{n-1} L_{5n(n+1)/2-n+2}} & \text{if } n \text{ is odd.} \end{cases}$$

As a consequence, we can compute the determinant of $\mathcal{W}_N$, since it is simply evaluated as $U_{1,1} \cdots U_{N,N}$ (we only state the Fibonacci version for $\lambda = 2$, $\mu = 3$, $r = 1$ and $s = -1$):

Theorem 5.5.

$$\det \mathcal{W}_N = (-1)^{\frac{1}{6}N(N^2-1)} \times \prod_{d=1}^{N} \frac{\Big(\prod_{t=1}^{d-1} F_{2t}\Big)\Big(\prod_{t=1}^{d-1} F_{3t}\Big)}{\Big(\prod_{t=1}^{d-1} L_{3t+2d-1}\Big)\Big(\prod_{t=1}^{d} L_{2t+3N-1}\Big)} \begin{cases} \frac{5^{d-1} L_{5(d+1)d/2-d+2}}{F_{5d(d-1)/2-d+3}} & \text{if } d \text{ is odd,} \\ \frac{5^d F_{5(d+1)d/2-d+2}}{L_{5d(d-1)/2-d+3}} & \text{if } d \text{ is even.} \end{cases}$$

Now we compute the inverse of the matrix $\mathcal{W}$. It depends on the dimension, so we compute $(\mathcal{W}_N)^{-1}$.

Theorem 5.6. *For $1 \leqslant d \leqslant n \leqslant N$:*

$$(\mathcal{W}_N)^{-1}_{n,d} = \mathbf{i}^{s-r}(-1)^{n-d-1+N} q^{\lambda\frac{d(d+1)}{2} + \mu\frac{n(n+1)}{2} - Ns - Nd\lambda - Nn\mu + \frac{r+s}{2}}$$

$$\times \frac{1}{(1+q^{s+\mu n+\lambda d})(1-q^{r-s})} \frac{(-q^{\lambda+\mu n+s};q^{\lambda})_N(-q^{\mu+\lambda d+s};q^{\mu})_N}{(q^{\lambda};q^{\lambda})_{N-d}(q^{\lambda};q^{\lambda})_{d-1}(q^{\mu};q^{\mu})_{n-1}(q^{\mu};q^{\mu})_{N-n}}$$
$$\times \frac{1+(-1)^N q^{(\lambda+\mu)\frac{N(N+1)}{2}-\lambda d-\mu n+r+sN-2s}}{1-(-1)^N q^{(\lambda+\mu)\frac{N(N+1)}{2}+r+sN-s}}. \qquad \blacksquare$$

Fibonacci Corollary for $\lambda=2$, $\mu=3$, $r=1$ and $s=-1$:

Corollary 5.5. *For* $1\leqslant d\leqslant n\leqslant N$*:*

$$(\mathcal{W}_N)^{-1}_{n,d} = \frac{(-1)^{\binom{n+1}{2}+n+d-Nn}\Big(\prod\limits_{t=1}^{N} L_{3t+2d-1}\Big)\Big(\prod\limits_{t=1}^{N} L_{2t+3n-1}\Big)}{\Big(\prod\limits_{t=1}^{d-1} F_{2t}\Big)\Big(\prod\limits_{t=1}^{n-1} F_{3t}\Big)\Big(\prod\limits_{t=1}^{N-d} F_{2t}\Big)\Big(\prod\limits_{t=1}^{N-n} F_{3t}\Big)L_{2d+3n-1}}$$
$$\times \begin{cases} \frac{L_{5N(N+1)/2-2d-3n-N+3}}{5^N F_{5N(N+1)/2-N+2}} & \textit{if } N \textit{ is even,}\\ \frac{F_{5N(N+1)/2-2d-3n-N+3}}{5^{N-1} L_{5N(N+1)/2-N+2}} & \textit{if } N \textit{ is odd.}\end{cases}$$

6. Proofs

Following our introductory remarks, we present now two proofs about the matrix $\mathcal{T}$.

In order to show that indeed $\mathcal{T}=L\cdot U$, we need to show that for any m, n:

$$\sum_d L_{m,d}U_{d,n} = t_{m,n} = \mathbf{i}^{1-r-\lambda m-\mu n} q^{\frac12(\lambda m+\mu n+r-1)}\frac{1-q}{1-q^{\lambda m+\mu n+r}}.$$

In rewritten form the formula to be proved reads

$$\sum_d L_{m,d}U_{d,n} = (q^{\lambda};q^{\lambda})_{m-1}(q^{\mu};q^{\mu})_{n-1}(1-q)\mathbf{i}^{1-r-\mu n-\lambda m}q^{\frac12(\lambda m+\mu n-r-1)}$$
$$\times \sum_d \frac{q^{\frac12 d^2(\lambda+\mu)+rd-\frac12\mu d-\frac{\lambda}{2}d}(q^{r+\mu+\lambda d};q^{\mu})_d}{(q^{\lambda};q^{\lambda})_{m-d}(q^{r+\mu+\lambda m};q^{\mu})_d(q^{\mu};q^{\mu})_{n-d}(q^{r+\lambda+\mu n};q^{\lambda})_d(q^{r+\mu+\lambda d};q^{\mu})_{d-1}}.$$

Apart form the constant factor, it should be proven that

$$\sum_d \frac{q^{\frac12 d^2(\lambda+\mu)+rd-\frac12\mu d-\frac{\lambda}{2}d}(q^{r+\mu+\lambda d};q^{\mu})_d}{(q^{\lambda};q^{\lambda})_{m-d}(q^{r+\mu+\lambda m};q^{\mu})_d(q^{\mu};q^{\mu})_{n-d}(q^{\lambda+\mu n+r};q^{\lambda})_d(q^{r+\mu+\lambda d};q^{\mu})_{d-1}}$$
$$= \frac{1}{(q^{\lambda};q^{\lambda})_{m-1}(q^{\mu};q^{\mu})_{n-1}(1-q^{\lambda m+\mu n+r})}. \tag{6.1}$$

For the verification of the last equation, let us denote the LHS of the equation (6.1) by $\mathtt{SUM}_m$, then the Mathematica version of the q-Zeilberger algorithm [**8**] produces the recursion

$$\mathtt{SUM}_m = \frac{1-q^{m(\lambda-1)+\mu n+r}}{(1-q^{m(\lambda-1)})(1-q^{m\lambda+\mu n+r})}\mathtt{SUM}_{m-1}.$$

Now we move to the inverse matrices. Since L and L^{-1} are lower triangular matrices, we only need to look at the entries indexed by (m,n) with $m \geqslant n$:

$$\sum_{n\leqslant d\leqslant m} L_{m,d}L^{-1}_{d,n} = \frac{(q^\lambda;q^\lambda)_{m-1}}{(q^\lambda;q^\lambda)_{n-1}}\mathbf{i}^{-\lambda m}q^{\frac12\lambda m}(-1)^n\mathbf{i}^{\lambda n}q^{\frac12\lambda n^2} \times \sum_{n\leqslant d\leqslant m}\frac{(-1)^d q^{-\lambda dn+\frac12\lambda d^2-\frac12\lambda d}(q^{\mu+\lambda d+r};q^\mu)_d(q^{\mu+\lambda n+r};q^\mu)_{d-1}}{(q^\lambda;q^\lambda)_{m-d}(q^{\mu+\lambda m+r};q^\mu)_d(q^\lambda;q^\lambda)_{d-n}(q^{\mu+\lambda d+r};q^\mu)_{d-1}}.$$

For the sum on d in the last expression, the q-Zeilberger algorithm evaluates it and give us 0 for $m \neq n$. For $m=n$, it is easy:

$$\frac{(q^\lambda;q^\lambda)_{n-1}}{(q^\lambda;q^\lambda)_{n-1}}\mathbf{i}^{-\lambda n}q^{\frac12\lambda n}(-1)^n\mathbf{i}^{\lambda n}q^{\frac12\lambda n^2} \times \frac{(-1)^n q^{-\lambda n^2+\frac12\lambda n^2-\frac12\lambda n}(q^{\mu+\lambda n+r};q^\mu)_n(q^{\mu+\lambda n+r};q^\mu)_{n-1}}{(q^{\mu+\lambda n+r};q^\mu)_n(q^{\mu+\lambda n+r};q^\mu)_{n-1}} = 1.$$

In that case, the equality is valid as well and so the proof is complete.

Now we present one proof for $\mathcal{M}$.

We start with an introductory remark. For all the identities that we need to prove, experiments indicate that they are Gosper-summable. However, the entries that we encounter in our instances, do not qualify for the q-Zeilberger algorithm that we used in our earlier papers. Therefore, it was necessary to *guess* the relevant quantities; the justification is then complete routine. However, this guessing procedure is (with all the parameters involved) extremely time consuming, and so we confined ourselves to the demonstration of one such proof. We hope that extensions of the q-Zeilberger algorithm will be developed that fit our needs.

We deal now with

$$\sum_{n\leqslant d\leqslant m} L_{m,d}L^{-1}_{d,n}$$

and prove that it is 1 for $n=m$ (there is only one term in the sum) and 0 for $n>m$ since we have lower triangular matrices. So let us assume $m>n$. We will prove a general formula depending on an extra variable K:

$$\sum_{n\leqslant d\leqslant K} L_{m,d}L^{-1}_{d,n} = (-1)^{K-n}q^{\lambda\frac{K(K+1)}{2}-\lambda Kn+\lambda\frac{n(n-1)}{2}} \times \frac{1-q^{(\lambda+\mu)\frac{K(K+1)}{2}+r+sK-s+(m-n)\lambda}}{1-q^{(\lambda+\mu)\frac{K(K+1)}{2}+r+sK-s}} \times \frac{1}{1-q^{\lambda m-\lambda n}}\frac{(q^\lambda;q^\lambda)_{m-1}}{(q^\lambda;q^\lambda)_{K-n}(q^\lambda;q^\lambda)_{m-1-K}(q^\lambda;q^\lambda)_{n-1}}\frac{(q^{s+\lambda n+\mu};q^\mu)_K}{(q^{s+\lambda m+\mu};q^\mu)_K}.$$

The formula we need follows from setting $K := m$. Note that the RHS of formula equals 0 when $K=m>n$ because of the term $(q^\lambda;q^\lambda)_{m-K-1}$ in the denominator of the second row. The proof of the formula is by induction. Clearly it is true for $K=n$, and the induction step amounts to show that

$$\sum_{n\leqslant d\leqslant K} L_{m,d}L_{d,n}^{-1} + L_{m,K+1}L_{K+1,n}^{-1} = \sum_{n\leqslant d\leqslant K+1} L_{m,d}L_{d,n}^{-1},$$

which equals

$$\begin{aligned}
&(-1)^{K-n} q^{\lambda\frac{K(K+1)}{2}-\lambda Kn+\lambda\frac{n(n-1)}{2}} \frac{1-q^{(\lambda+\mu)\frac{K(K+1)}{2}+sK-s+r+(m-n)\lambda}}{1-q^{(\lambda+\mu)\frac{K(K+1)}{2}+sK-s+r}} \frac{1}{1-q^{\lambda m-\lambda n}}\\
&\quad\times \frac{(q^\lambda;q^\lambda)_{m-1}}{(q^\lambda;q^\lambda)_{K-n}(q^\lambda;q^\lambda)_{m-1-K}(q^\lambda;q^\lambda)_{n-1}} \frac{(q^{s+\lambda n+\mu};q^\mu)_K}{(q^{s+\lambda m+\mu};q^\mu)_K}\\
&\quad+ \frac{(q^\lambda;q^\lambda)_{m-1}}{(q^\lambda;q^\lambda)_K(q^\lambda;q^\lambda)_{m-K-1}} \frac{(q^{s+\lambda(K+1)+\mu};q^\mu)_{K+1}}{(q^{s+\lambda m+\mu};q^\mu)_{K+1}}\\
&\quad\times \frac{1-q^{(\lambda+\mu)\frac{(K+1)(K+2)}{2}+\lambda(m-K-1)+sK+r}}{1-q^{(\lambda+\mu)\frac{(K+1)(K+2)}{2}+sK+r}}\\
&\quad\times (-1)^{K+1-n} q^{\lambda\frac{K(K+1)}{2}+\lambda\frac{n(n+1)}{2}-\lambda n(K+1)} \frac{(q^{s+\lambda n+\mu};q^\mu)_K}{(q^{s+\lambda(K+1)+\mu};q^\mu)_K}\\
&\quad\times \frac{1-q^{-\lambda n+\frac{\lambda+\mu}{2}(K+1)^2+\frac{\lambda-\mu}{2}(K+1)+r+s(K-1)}}{1-q^{-\lambda(K+1)+\frac{\lambda+\mu}{2}(K+1)^2+\frac{\lambda-\mu}{2}(K+1)+r+s(K-1)}} \frac{(q^\lambda;q^\lambda)_K}{(q^\lambda;q^\lambda)_{n-1}(q^\lambda;q^\lambda)_{K+1-n}}\\
&\quad= (-1)^{K+1-n} q^{\lambda\frac{(K+1)(K+2)}{2}-\lambda n(K+1)+\lambda\frac{n(n-1)}{2}}\\
&\quad\times \frac{1-q^{(\lambda+\mu)\frac{(K+1)(K+2)}{2}+s(K+1)-s+r+(m-n)\lambda}}{1-q^{(\lambda+\mu)\frac{(K+1)(K+2)}{2}+s(K+1)-s+r}} \frac{1}{1-q^{\lambda m-\lambda n}}\\
&\quad\times \frac{(q^\lambda;q^\lambda)_{m-1}}{(q^\lambda;q^\lambda)_{K-n+1}(q^\lambda;q^\lambda)_{m-K-2}(q^\lambda;q^\lambda)_{n-1}} \frac{(q^{s+\lambda n+\mu};q^\mu)_{K+1}}{(q^{s+\lambda m+\mu};q^\mu)_{K+1}},
\end{aligned}$$

or

$$\begin{aligned}
&-(1-q^{(\lambda+\mu)\frac{K(K+1)}{2}+sK-s+r+(m-n)\lambda})(1-q^{(\lambda+\mu)\frac{(K+1)(K+2)}{2}+sK+r})\\
&\quad\times(1-q^{\lambda(K-n+1)})(1-q^{s+\lambda m+\mu+\mu K})\\
&+(1-q^{s+\lambda(K+1)+\mu+\mu K})\times(1-q^{(\lambda+\mu)\frac{(K+1)(K+2)}{2}+\lambda(m-K-1)+sK+r})\\
&\quad\times(1-q^{-\lambda n+\frac{\lambda+\mu}{2}(K+1)^2+\frac{\lambda-\mu}{2}(K+1)+r+s(K-1)})(1-q^{\lambda m-\lambda n})\\
&=q^{\lambda+K\lambda-n\lambda}(1-q^{(\lambda+\mu)\frac{(K+1)(K+2)}{2}+s(K+1)-s+r+(m-n)\lambda})\\
&\quad\times(1-q^{(\lambda+\mu)\frac{K(K+1)}{2}+sK-s+r})(1-q^{\lambda(m-K-1)})(1-q^{s+\lambda n+\mu+\mu K}),
\end{aligned}$$

which is a routine check. Thus we have the claimed result.

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