HOW TO SELECT A LOSER

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ABSTRACT. N people select a *loser* by flipping coins. Recursively, the 0-party continues until the loser is found. Among other things, it is shown that this process stops on the average after about $\log_2 N$ steps. Nevertheless, this very plausible result requires rather advanced methods.

1. INTRODUCTION

Assume that a party of N people wants to select one of their members (the *loser*) e.g. in order to pay a beer for everybody. The procedure is as follows: Everybody is flipping a coin (with outputs 0 and 1, each with probability $\frac{1}{2}$); then, recursively, the 0-party is selecting one of their members. However, there is one exception: If all people have thrown a "1", then they have to repeat the procedure.

The following example illustrates the idea.

Fig.1

What is produced by the above mentioned mechanism might be called an *incomplete trie*. In this figure a left branch is labelled by a *0* and a right branch by a *1*.

For tries and related data structures we refer to [4]. The idea of splitting the party by flipping a coin was also used in the *tree protocol* of Capetanakis and Tsybakhov (see [5] for a survey).

In the following we analyze the *size* of the tree, i.e the number of nodes. (In the example the size is 8.) It will turn out that the average size is about $2\log_2 N$; for a more precise statement see the following sections.

Another parameter of interest is the number of times the party has to flip the coins (= the *depth*). (In the example this number is 4.)

Also, we consider the total number of coin flippings. (In the example this number is 28.)

It is worthwhile to mention that, even though the problem seems to be very simple, the analysis is not totally trivial.

2. Preliminaries

Here we list some notational conveniences, as well as some basic properties of the Bernoulli numbers, since they are needed in the sequel (compare e.g. [3]).

We write $[z^n]f(z)$ for the coefficient of z^n in the series $f(z)$. Quite naturally we have

$$\left[\frac{z^n}{n!}\right] f(z) = n!\,[z^n]\,f(z).$$

Abbreviations:

$$L = \log 2 \qquad \text{and} \qquad \chi_k = \frac{2k\pi i}{L}, \quad k \in \mathbb{Z}.$$

$\zeta(s)$ denotes Riemann's ζ–function; $\Gamma(s)$ the Γ–function; γ is Euler's constant; $\gamma = 0.57721$.

The Bernoulli numbers B_n are defined by

$$B_n = \left[\frac{z^n}{n!}\right] \frac{z}{e^z - 1}.$$

Similarly, the Bernoulli polynomial $B_n(t)$ is defined by

$$\frac{ze^{tz}}{e^z - 1} = \sum_{k\ge 0} B_k(t)\frac{z^k}{k!}.$$

Properties:

$$B_n(0) = B_n(1) = B_n \quad \text{for } n \ge 2;\ B_0 = 1,\ B_1 = -\frac{1}{2},$$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k\, x^{n-k}$$

and thus

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k \quad \text{and} \quad B_n\left(\frac{1}{2}\right) = -\left(1 - 2^{1-n}\right) B_n .$$

Also, we may express the Bernoulli numbers by the ζ–function:

$$B_k = -k\zeta(1-k) \quad \text{for } k \geq 2.$$

3. The average size

The style of the following analysis follows [4]; we mention also [6].

Proposition 1. *The probability generating function $F_N(z)$ fulfills the following recursion:*

$$\begin{aligned} F_N(z) &= z^2 \sum_{k=0}^{N} 2^{-N} \binom{N}{k} F_k(z) + z(1-z)2^{-N} F_N(z) \\ &\quad -z^2 2^{-N} + z2^{-N} F_N(z) \text{ for } N \geq 2; \ F_0(z) = 1, \ F_1(z) = z. \end{aligned}$$

Proof. $2^{-N}\binom{N}{k}$ is the probability that k out of N people have thrown a 0; z^2 measures two extra nodes obtained by this splitting. The extra terms are related to the cases $k = 0$ or $k = N$. In these two cases there is only one extra node generated, which is measured by a z.

Let $l_N = F_N'(1)$ denote the average size. The following recursion follows immediately from Proposition 1 by differentiation:

Proposition 2. *We have $l_0 = 0$, $l_1 = 1$ and for $N \geq 2$*

$$l_N = 2 + \frac{1}{1 - 2^{-N}} \sum_{k=0}^{N} 2^{-N} \binom{N}{k} l_k .$$

Now we introduce the *exponential generating function* $L(z) = \sum_{k \geq 0} l_k \frac{z^k}{k!}$. Then the recursion for the l_n's turns into a functional equation for $L(z)$.

Proposition 3.

$$L(z) - L\left(\frac{z}{2}\right) = 2e^z - 2e^{z/2} - z + e^{z/2} L\left(\frac{z}{2}\right).$$

The equation becomes easier by introducing

$$\hat{L}(z) = \frac{L(z)}{e^z - 1} = \sum_{k \geq 0} \hat{l}_k \frac{z^k}{k!}.$$

Proposition 4.

$$\hat{L}(z) = \hat{L}(\frac{z}{2}) + \frac{2e^z - 2e^{z/2} - z}{e^z - 1}.$$

Proposition 5. *$\hat{l}_0 = 1$ and for $N \geq 1$ we have*

$$\hat{l}_N = \frac{1}{1-2^{-N}} \frac{2}{N+1} B_{N+1} + \frac{2}{N+1} B_{N+1} - \frac{B_N}{1-2^{-N}}.$$

Proof. From Proposition 4 we find that for $N \geq 1$

$$\begin{aligned}
\hat{l}_N(1-2^{-N}) &= \left[\frac{z^N}{N!}\right] \frac{2e^z - 2e^{z/2} - z}{e^z - 1} \\
&= \frac{2}{N+1} \left[\frac{z^{N+1}}{(N+1)!}\right] \frac{ze^z}{e^z - 1} \\
&- \frac{2}{N+1} \left[\frac{z^{N+1}}{(N+1)!}\right] \frac{ze^{z/2}}{e^z - 1} - B_N \\
&= \frac{2}{N+1} B_{N+1}(1) - \frac{2}{N+1} B_{N+1}(\frac{1}{2}) - B_N.
\end{aligned}$$

The final formula follows from the properties of the Bernoulli numbers.

Now, since $L(z) = \big(e^z - 1\big)\hat{L}(z)$, we find

$$\begin{aligned}
l_N &= \sum_{k=0}^{N-1} \binom{N}{k} \hat{l}_k \\
&= 1 + \sum_{k=1}^{N-1} \binom{N}{k} \frac{1}{1-2^{-k}} \frac{2}{k+1} B_{k+1} \\
&+ \sum_{k=1}^{N-1} \binom{N}{k} \frac{2}{k+1} B_{k+1} - \sum_{k=1}^{N-1} \binom{N}{k} \frac{B_k}{1-2^{-k}} \\
&= 1 + S_1 + S_2 + S_3.
\end{aligned}$$

The sum S_2 is the easiest:

$$\begin{aligned}
S_2 &= \frac{2}{N+1} \sum_{k=1}^{N-1} \binom{N+1}{k+1} B_{k+1} \\
&= \frac{2}{N+1} \left[\sum_{k=0}^{N+1} \binom{N+1}{k} B_k - B_{N+1} - (N+1)B_1 - B_0 \right] \\
&= 1 - \frac{2}{N+1} \sim 1.
\end{aligned}$$

Next we turn to S_1:

$$S_1 = \sum_{k=1}^{N-1} \binom{N}{k} \frac{2}{k+1} B_{k+1} \left(1 + \frac{1}{2^k - 1}\right)$$
$$= S_2 + \frac{2}{N+1} \sum_{k=2}^{N} \binom{N+1}{k} \frac{B_k}{2^{k-1} - 1}.$$

The remaining sum was studied in Knuth [4, p.503]:

Theorem 6. [Knuth]

$$S := \frac{1}{n} \sum_{k=2}^{n-1} \binom{n}{k} \frac{B_k}{2^{k-1} - 1} \sim \frac{1}{2} \log_2 n - \frac{1}{2} \log_2 \pi + \frac{\gamma}{2L} - \frac{3}{4} + \delta_1(\log_2 n),$$

with the periodic function (of period 1 and very small amplitude)

$$\delta_1(x) = \frac{1}{L} \sum_{k \neq 0} \zeta(-\chi_k) \Gamma(-\chi_k) e^{2k\pi i x}. \quad \square$$

With this result we can formulate an asymptotic equivalent for the sum S_1:

$$S_1 \sim \log_2 N - \log_2 \pi + \frac{\gamma}{L} - \frac{1}{2} + 2\delta(\log_2 N).$$

Now we turn to the computation of the last sum, S_3:

$$S_3 = -\sum_{k=1}^{N-1} \binom{N}{k} B_k \left[1 + \frac{1}{2^k - 1}\right].$$

According to this decomposition we find that the first sum equals 1 and analyze the second one:

Theorem 7.

$$S := \sum_{k=1}^{n-1} \binom{n}{k} \frac{B_k}{2^k - 1} \sim -\log_2 n + \frac{1}{2} + \delta_2(\log_2 n),$$

with the periodic function (of period 1 and very small amplitude)

$$\delta_2(x) = \frac{1}{L} \sum_{k \neq 0} \zeta(1 - \chi_k) \Gamma(1 - \chi_k) e^{2k\pi i x}.$$

Proof. Instead of following Knuth's approach (cf.Theorem 6) (Mellin transforms) we use "Rice's method" (see [1] for a good introduction). An equivalent methodology was given by W. Szpankowski in [7]. From this, we can express the sum S as a contour integral

$$S = \frac{1}{2\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \frac{(-1)^n n!}{(z-1)(z-2)\dots(z-n)} \cdot \frac{\zeta(1-z)}{2^z - 1} dz.$$

Shifting the line of integration to the left and collecting the residues gives the asymptotic expansion. The dominant poles are at $z=0$ and $z=\chi_k$, $k\in\mathbb{Z}$, $k\neq 0$.

The local expansions for $z\to 0$ are

$$\frac{(-1)^n n!}{(z-1)(z-2)\dots(z-n)} \sim 1+zH_n$$
$$\zeta(1-z)\sim -\frac{1}{z}(1-\gamma z)$$
$$\frac{1}{2^z-1}\sim\frac{1}{Lz}\left(1-\frac{Lz}{2}\right)$$

with $H_n=1+\frac{1}{2}+\dots+\frac{1}{n}\sim\log n+\gamma$ being the n–th harmonic number. The residue at $z=0$ is therefore

$$\begin{aligned}&\left[z^{-1}\right]-\frac{1}{Lz^2}(1+zH_n)(1-\gamma z)\left(1-\frac{Lz}{2}\right)\\&=-\frac{1}{L}\left(H_n-\gamma-\frac{L}{2}\right)\\&\sim-\log_2 n+\frac{1}{2}.\end{aligned}$$

The residue at $z=\chi_k$ is $\frac{1}{L}\Gamma(1-\chi_k)\zeta(1-\chi_k)$. This may be seen by expressing

$$\frac{(-1)^n n!}{(z-1)(z-2)\dots(z-n)}\quad\text{as}\quad\frac{\Gamma(n+1)\Gamma(1-z)}{\Gamma(n+1-z)}.$$

So the function $\delta_2(x)$ is obtained, since $n^{\chi_k}=e^{2k\pi i\cdot\log_2 n}$.

Therefore we find that the sum S_3 can be approximated by

$$S_3\sim\log_2 n+\frac{1}{2}-\delta_2(\log_2 N).$$

Now we get our main result by collecting $1+S_1+S_2+S_3$:

Theorem 8. *The average size l_N of the tree built by N people who are selecting a loser by a coin flipping process is*

$$l_N\sim 2\log_2 N-\log_2\pi+\frac{\gamma}{L}+2+2\delta_1(\log_2 N)-\delta_2(\log_2 N).$$

The constant $-\log_2\pi+\frac{\gamma}{L}+2$ *is* $=1.181250048$.

4. The average depth

The analysis of the depth is a little bit easier than the size and almost included in the computations from Section 3. Especially, as was observed in [6], the methodology to solve recursions as in Section 3 is general enough to deal with all the following ones. We just state the analogous propositions:

Proposition 9. *The probability generating function $G_N(z)$ fulfills for $N \geq 2$ ($G_0(z) = G_1(z) = 1$):*

$$G_N(z) = z\sum_{k=0}^{N} 2^{-N}\binom{N}{k} G_k(z) - z2^{-N} + z2^{-N}G_N(z).$$

Set $d_N = G'_N(1)$.

Proposition 10. *We have $d_0 = d_1 = 0$ and for $N \geq 2$:*

$$d_N\left(1 - 2^{-N}\right) = 1 + \sum_{k=0}^{N} 2^{-N}\binom{N}{k} d_k.$$

Set

$$D(z) = \sum_{k\geq 0} d_k \frac{z^k}{k!} \quad \text{and} \quad \hat{D}(z) = \frac{D(z)}{e^z - 1} = \sum_{k\geq 0} \hat{d}_k \frac{z^k}{k!}.$$

Proposition 11.

$$D(z) - D\left(\frac{z}{2}\right) = e^z - 1 - z + e^{z/2} D\left(\frac{z}{2}\right)$$

Proposition 12.

$$\hat{D}(z) = \hat{D}\left(\frac{z}{2}\right) + 1 - \frac{z}{e^z - 1}.$$

Proposition 13. *$\hat{d}_0 = 1$ and for $N \geq 1$ we have*

$$\hat{d}_N = -\frac{B_N}{1 - 2^{-N}}.$$

Therefore we find that d_N equals S_3 from Section 1 and we have:

Theorem 14. *The average depth d_N of the tree built by N people who are selecting a loser by a coin flipping process is*

$$d_N \sim \log_2 N + \frac{1}{2} - \delta_2(\log_2 N).$$

5. The average number of coin flippings

Now we count the total number of coin flippings during the process; if each of the N people flips a coin, this contributes N to the total number. Surprisingly there is an easy exact formula for the average:

Theorem 15. *The average total number c_N of coin flippings performed by N people who are selecting a loser is*

$$c_N = 2N \quad \text{for } N \ge 2, \quad c_0 = c_1 = 0.$$

Proof. We use some obvious notations in the style of the preceeding sections.

$$F_N(z) = z^N \sum_{k=0}^{N} 2^{-N} \binom{N}{k} F_k(z) - z^N 2^{-N} + z^N 2^{-N} F_N(z)$$

$$\text{for } N \ge 2, \quad F_0(z) = F_1(z) = 1.$$

$$c_N \left(1 - 2^{-N}\right) = N + \sum_{k=0}^{N} 2^{-N} \binom{N}{k} c_k \quad \text{for } N \ge 2, \quad c_0 = c_1 = 0.$$

$$C(z) - C\Big(\frac{z}{2}\Big) = z\left(e^z - 1\right) + e^{z/2} C\Big(\frac{z}{2}\Big).$$

$$\hat{C}(z) = z + \hat{C}\Big(\frac{z}{2}\Big).$$

$$\hat{c}_N \left(1 - 2^{-N}\right) = 0 \quad \text{for } N \ne 1, \quad \hat{c}_1 = 2.$$

$$c_N = \sum_{k=0}^{N-1} \binom{N}{k} \hat{c}_k = 2N.$$

6. A draw is possible

We might think about stopping the process if exactly 2 people are fighting about being the loser (because it is unfair; they should share the costs for the beers!). In this section we consider the averages from before in this model. We use the notations from the preceeding sections, but with the slightly different meaning.

We need yet another asymptotic formula involving the Bernoulli numbers:

Theorem 16.

$$S := n \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k}{2^{k+1} - 1} \sim \frac{\pi^2}{6L} + \delta_3(\log_2 n),$$

with the periodic function (of period 1 and very small amplitude)

$$\delta_3(x) = \frac{1}{L} \sum_{k \ne 0} \zeta(2 - \chi_k) \Gamma(2 - \chi_k) e^{2k\pi i x}.$$

Proof. As in the proof of Theorem 7, we express S as a contour integral:

$$S = n \cdot \frac{1}{2\pi i} \int_{-\frac{1}{2} - i\infty}^{-\frac{1}{2} + i\infty} \frac{(-1)^n n!}{(z-1)(z-2)\dots(z-n)} \cdot \frac{\zeta(1-z)}{2^{z+1} - 1} dz.$$

The residues at $z = -1$ and $z = -1 + \chi_k$, $k \ne 0$, give the asymptotic formula (observe that $\zeta(2) = \dfrac{\pi^2}{6}$).

Theorem 17.

$$d_N \sim \log_2 N + \frac{1}{2} - \frac{\pi^2}{12L} - \delta_2(\log_2 N) - \frac{1}{2}\delta_3(\log_2 N).$$

The constant $\frac{\pi^2}{12L} = 1.1865691$ *describes how much is saved by stopping earlier.*

Proof. It turns out that

$$\hat{d}_N = \frac{1}{1-2^{-N}}\left(-B_N - \frac{1}{2}NB_{N-1}\right).$$

Therefore

$$d_N = 1 - \sum_{k=1}^{N-1}\binom{N}{k}\frac{B_k}{2^k-1} - \frac{N}{2}\sum_{k=0}^{N-2}\binom{N-1}{k}\frac{B_k}{2^{k+1}-1}.$$

So the result follows from the Theorems 6 and 7.

Theorem 18.

$$\begin{aligned} l_N &\sim 2\log_2 N - \log_2 \pi + \frac{\gamma}{L} + 2 - \frac{\pi^2}{16L} \\ &\quad + 2\delta_1(\log_2 n) - \delta_2(\log_2 n) - \frac{3}{8}\delta_3(\log_2 N). \end{aligned}$$

The constant $\frac{\pi^2}{16L}$ *is* $= 0.88992683$.

Proof. The old $\hat{l}_N$ must be changed by the additive term $-\frac{3}{8}\frac{NB_{N-1}}{1-2^{-N}}$. Therefore l_N changes by

$$-\frac{3}{8}\sum_{k=1}^{N-1}\binom{N}{k}\frac{kB_{k-1}}{1-2^{-k}},$$

which is covered by Theorem 16.

Theorem 19.

$$c_N \sim 2N - \frac{\pi^2}{6L} - \delta_3(\log_2 N).$$

Proof. The extra term is again covered by the formula in Theorem 16.

Remark. There is a slightly more general formula involving the Bernoulli numbers (with an identical proof):

Theorem 20. *Let $a > 0$ be an arbitrary real number. Then*

$$\begin{aligned} n^a\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_k}{2^{k+a}-1} &\sim \frac{1}{L}\zeta(1+a)\Gamma(1+a) \\ &\quad + \frac{1}{L}\sum_{k\neq 0}\zeta(a-\chi_k)\Gamma(a-\chi_k)e^{2k\pi ix}. \end{aligned}$$

7. Compression of the tree ("Patricia")

If we don't create a new node in the case that all party members have thrown the same side of the coin, we construct a *compressed (incomplete) trie.* In Computer Science it is called a *Patricia trie* [4].

We just mention by how much the averages from the Sections 3–5 are lowered in this way. (Only the main term is mentioned, and also none of the (small) fluctuations.) The computations are totally similar to the earlier ones.

$$\begin{aligned} \text{size} \;\; l_N &: \quad \log_2 \pi - \frac{\gamma}{L} = 0.81874995 \\ \text{depth} \;\; d_N &: \quad \log_2 \pi - \frac{\gamma}{L} = 0.81874995 \\ \text{flippings} \;\; c_N &: \quad \frac{N}{2} \end{aligned}$$

8. Further research

The computation of the variances in all instances; this seems to be difficult.

Selection of b losers: Recursively, the 0-party (k members) is looking for the b losers. However, if $k < b$, k losers are already found, so the 1-party must select the remaining $b - k$ losers! There are a lot of every day situations were this principle might apply. The reader is invited to formulate some stimulating examples.

A reasonable variation is that not automatically the 0-party is looking (recursively) for the loser, but always the party with *smaller cardinality* (with some convention if they have the same number of elements).

We hope to report on these and other problems in the near future.

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