ASYMPTOTICS OF THE LURIA-DELBRÜCK DISTRIBUTION

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We consider the Luria–Delbrück distribution, as described in [2] and previous writings, cited therein. It is given by the probability generating function

$$G(z) = (1-z)^{m(1-z)/z} = \sum_{n\ge 0} p_n z^n,$$

where m is a parameter.

We show that with *singularity analysis of generating functions* [1] (in principle) a full asymptotic expansion of p_n and $P_n := \sum_{k>n} p_k$ can be achieved. In this way we generalize results given in [2].

The basic idea of this powerful method is to realize that $G(z)$ has radius of convergence 1 and 1 as the only singularity on the radius of convergence. Then there is a *local expansion* about $z=1$ (produced by Maple),

$$\begin{aligned} G(z) &= 1 - m(1-z)\log\frac{1}{1-z} - m(1-z)^2\log\frac{1}{1-z} \\ &\quad + \frac{m^2}{2}(1-z)^2\log^2\frac{1}{1-z} + \mathcal{O}\left((1-z)^3\log^3\frac{1}{1-z}\right), \end{aligned} \tag{1}$$

which can be translated *term–by–term* into an asymptotic expansion of the coefficients under some mild technical conditions which are satisfied in out instance; we refer again to [1] for a full treatment.

Therefore we only have to know about the expansion of the standard functions appearing in (1).

We use the following notions: $[z^n]f(z)$ means the coefficient of z^n in (the Taylor expansion of) $f(z)$, the symbol $n^{\underline{k}}$ means $n(n-1)\dots(n-k+1)$ and $H_n = 1+\frac12+\frac13+\dots+\frac1n$ means the nth harmonic number. Then we have for n large enough

$$\begin{aligned} [z^n](1-z)^i\log\frac{1}{1-z} &= \frac{(-1)^i i!}{n^{\underline{i+1}}}, \\ [z^n]\log^2\frac{1}{1-z} &= \frac{2}{n}H_{n-1}, \\ [z^n](1-z)^2\log^2\frac{1}{1-z} &= \frac{2}{n}H_{n-1} - \frac{4}{n-1}H_{n-2} + \frac{2}{n-2}H_{n-3}. \end{aligned}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Hence

$$p_n = \frac{m}{n^2} + \frac{2m^2}{n^3} + \frac{m^2}{2}\left(\frac{2}{n}H_{n-1} - \frac{4}{n-1}H_{n-2} + \frac{2}{n-2}H_{n-3}\right) + \mathcal{O}\left(\frac{\log^2 n}{n^4}\right).$$

MAPLE also translates this expansion into more traditional terms,

$$p_n = \frac{m}{n^2} + 2m^2\frac{\log n}{n^3} + \frac{2m^2\gamma - 3m^2 - m}{n^3} + \mathcal{O}\left(\frac{\log^2 n}{n^4}\right).$$

To get the asymptotic expansions for the tails P_n, we just note that its generating function is

$$\begin{aligned}\frac{1}{1-z} - \frac{1}{1-z}G(z) &= m\log\frac{1}{1-z} + m(1-z)\log\frac{1}{1-z}\\ &\quad - \frac{m^2}{2}(1-z)\log^2\frac{1}{1-z} + \mathcal{O}\left((1-z)^2\log^3\frac{1}{1-z}\right),\end{aligned}$$

from which we conclude

$$\begin{aligned}P_n &= \frac{m}{n} - \frac{m}{n^2} - \frac{m^2}{2}\left(\frac{2}{n}H_{n-1} - \frac{2}{n-1}H_{n-2}\right) + \mathcal{O}\left(\frac{\log^2 n}{n^3}\right)\\ &= \frac{m}{n} + m^2\frac{\log n}{n^2} + \frac{m^2\gamma - m^2 - m}{n^2} + \mathcal{O}\left(\frac{\log^2 n}{n^3}\right).\end{aligned}$$

REFERENCES

1. P. Flajolet and A. Odlyzko, *Singularity Analysis of Generating Functions*, SIAM J.Discrete Mathematics **3** (1990), 216–240.
2. A.W. Kemp, *Comments on the Luria–Delbrück Distribution*, J. Appl. Prob. **31** (1994), 822-828.

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