Combinatorial Sums in the Analysis of a Generalized van der Pol Equation

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Introduction

In [1, 2, 3] I found the following sums, all of them unevaluated. In this little note I will demonstrate how they can be computed with the help of MAPLE.

Before we start, let me say a few words about the origin of the problem: The generalized van der Pol equation is

$$\frac{d^2u}{dt^2}+u=\varepsilon(1-u^{2n})\frac{du}{dt}. \tag{1}$$

In order to obtain a *perturbation solution*, a first order equation (in ε)

$$\sin T\frac{dF_1}{dT}+2\cos TF_1=(1-a_0^{2n}\cos^{2n}T)\sin T \tag{2}$$

was studied; the solution involves the sums that will be discussed in the sequel.

For $1\le k\le n$, set

$$B_k^{(1)}(n):=-\frac{k}{\binom{2n+1}{n}}\sum_{j=k}^{n}\frac{2j+1}{j(j+1)}\binom{2n+1}{n-j}\ . \tag{3}$$

The first sum [1, (3.18)]:

$$\mathrm{SUM}_1:=B_1^{(1)}(n)=\ ? \tag{4}$$

The second sum [1, (4.11)]:

$$\mathrm{SUM}_2:=\sum_{k=1}^{n}\frac{1}{k}B_k^{(1)}(n)+\frac{B_1^{(1)}(n)}{2}=\ ? \tag{5}$$

The third sum [1, (5.14)]:

$$\mathrm{SUM}_3:=\frac{1}{\binom{2n+1}{n}^2}\sum_{k=1}^{n}\frac{(2k+1)^2}{k(k+1)}\binom{2n+1}{n-k}^2=\ ? \tag{6}$$

The fourth sum [3]:

$$\mathrm{SUM}_4:=\sum_{k=1}^{n}\binom{2n+2}{n+1-k}^2\left[k^2(4n+1)-n(n+1)\right]=\ ? \tag{7}$$

In particular, it was conjectured that

$$\begin{aligned}(n+1)^2\binom{2n+1}{n}^2\mathrm{SUM}_3&=(n+1)(2n+1)^3\\&-(4n+1)(n+1)^2\binom{2n+1}{n}^2\\&+(2n+1)\mathrm{SUM}_4.\end{aligned} \tag{8}$$

Results

Let $H_n=\sum_{k=1}^{n}\frac{1}{k}$ denote the nth harmonic number.

I found the following.

Theorem 1

$$\mathrm{SUM}_1=1-H_{n+1}\ . \tag{9}$$

Theorem 2

$$\mathrm{SUM}_2=\frac{3}{2}-\frac{1}{2(n+1)}-\frac{4^n}{\binom{2n}{n}}\ . \tag{10}$$

Theorem 3

$$\mathrm{SUM}_3=\frac{\binom{4n+1}{2n}}{\binom{2n}{n}^2}-3+\frac{2}{n+1}\ . \tag{11}$$

Theorem 4

$$\begin{aligned}\mathrm{SUM}_4&=\binom{4n+4}{2n+2}\frac{(n+1)(2n+1)}{2(4n+3)}\\&+\frac{(n+1)n}{2}\binom{2n+2}{n+1}^2-(n+1)(2n+1)^2\ .\end{aligned} \tag{12}$$

Sketch of Human Proofs

Here I will indicate how one would evaluate the sums without a computer.

To deal with SUM_1, define

$$f_n=\sum_{k=1}^{n}\frac{2k+1}{k(k+1)}\binom{2n+1}{n-k} \tag{13}$$

and note that

$$\binom{2n-1}{n-1-k}=\binom{2n+1}{n-k}\frac{n(n+1)-k(k+1)}{(2n+1)(2n)}, \tag{14}$$

from which we find

$$f_{n-1}=\frac{n(n+1)}{(2n+1)(2n)}f_n-\frac{1}{2(n+1)}\binom{2n}{n}. \tag{15}$$

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Introducing *summation factors*, this first order recurrence is solved by summing.

To deal with SUM_2, we start from

$$\frac{1}{k}B_k^{(1)}(n) = 1 - H_{n+1} + \frac{1}{\binom{2n+1}{n}}\sum_{j=1}^{k-1}\frac{2j+1}{j(j+1)}\binom{2n+1}{n-j}. \tag{16}$$

Summing on k and interchanging the order of summation, we get

$$\sum_{k=1}^{n}\frac{1}{k}B_k^{(1)}(n) = n\,(1-H_{n+1}) \tag{17}$$
$$+\frac{1}{\binom{2n+1}{n}}\sum_{1\le j\le n}\frac{2j+1}{j(j+1)}\binom{2n+1}{n-j}(n-j).$$

The remaining sum is handled in the style of SUM_1.

For SUM_3, we use that

$$\frac{(2k+1)^2}{k(k+1)} = 4 + \frac{1}{k} - \frac{1}{k+1} \tag{18}$$

and

$$\binom{2n+1}{n-k}^2 - \binom{2n+1}{n+1-k}^2 = -\frac{k}{n+1}\binom{2n+2}{n+1-k}^2 \tag{19}$$

to rewrite the sum as

$$\mathrm{SUM}_3 = 2\frac{\binom{4n+2}{2n+1}}{\binom{2n+1}{n}^2} - \frac{1}{(n+1)\binom{2n+1}{n}^2}\sum_{k=1}^{n+1}\binom{2n+2}{n+1-k}^2 - 3; \tag{20}$$

the remaining evaluation is standard (Vandermonde, see [4]), as is the evaluation of SUM_4.

Computer Proofs

- SUM_1

We use the MAPLE program 'ekhad', by Doron Zeilberger. (Later versions of MAPLE should include similar programs in the *sumtools* package)

```
> read `ekhad`:
```

Version of Oct. 17, 1995

This is EKHAD, One of the Maple packages

accompanying the forthcoming book

"A = B"

(soon to be published by A.K. Peters, Ltd.)

by Marko Petkovsek, Herb Wilf, and Doron Zeilberger.

The most current version of EKHAD is always

available by anonymous ftp to

ftp.math.temple.edu/pub/zeilberg/programs/EKHAD

or on the www : http : //www.math.temple.edu/~zeilberg

For general help, and a list of the available functions,

type "ezra();"

For specific help type "ezra(procedure_name)" .

```
> ezra();
```

A Maple package for proving Hypergemetric

(Binomial Coeff.) and other kinds of identities .

For help with a specific procedure,

type "ezra(procedure_name);"

Contains procedures :

findrec, ct, zeil, zeilpap, zeillim,

AZd, AZc, AZpapd, AZpapc, celine.

Now we define the general summand in SUM_1 to be $F = F(n,k)$,

```
> F:=-1/binomial(2*n+1,n)
> *(2*k+1)/k/(k+1)*binomial(2*n+1,n-k);
```

$$F := -\frac{(2\,k+1)\,\mathrm{binomial}(\,2\,n+1,n-k\,)}{\mathrm{binomial}(\,2\,n+1,n\,)\,k\,(\,k+1\,)}$$

and use Zeilberger's algorithm;

```
> zeil(F,k,n);
```

$$1-N,\frac{k\,(\,k+1\,)}{(\,2\,k+1\,)\,(\,n-k+1\,)}$$

The answer means the following: If we set

$$R(n,k) := \frac{k(k+1)}{(2k+1)(n-k+1)}$$

and

$$G(n,k) := F(n,k)\,R(n,k),$$

then

$$F(n,k) - F(n+1,k) = G(n,k+1) - G(n,k). \tag{21}$$

We might sum this up about $k \ge 1$ and get

$$\mathrm{SUM}_1(n) - \mathrm{SUM}_1(n+1) = -G(n,1) = \frac{1}{n+2}. \tag{22}$$

Unwinding this recursion, we get the result of Theorem 1.

• SUM_2

We start from equation (17) and have to prove

$$F(n) := \frac{1}{\binom{2n+1}{n}} \sum_{1\le k\le n} \frac{2k+1}{k(k+1)} \binom{2n+1}{n-k} (n-k) = 1 + \frac{H_n}{2} - n + nH_{n+1} - \frac{4^n}{\binom{2n}{n}} \,. \tag{23}$$

This formula might be (provided, we don't 'know' it already) *guessed* with GFUN, a MAPLE package written by Bruno Salvy and Paul Zimmermann, [5]:

```
> read `gfun`: with(gfun):
> l:=[-3/4,-4/3,-73/40,-79/35,-667/252,
>     -694/231,-22901/6864,...];
> guessgf(l,z);
```

I did not really run this particular example, since it would require a larger computer than I currently have at my disposal.

We apply Zeilberger's algorithm as before with

```
> F:=1/binomial(2*n+1,n)*(2*k+1)
> *(k*(k+1)*binomial(2*n+1,n-k)*(n-k):
> zeil(F,k,n);
```

and find

$$\begin{aligned}&2\,(\,n+1\,)\,(\,2\,n+3\,) - (\,4\,n+7\,)\,N\,(\,2\,n+1\,)\\ &\quad + N^2\,(\,2\,n+3\,)\,(\,2\,n+1\,),\\ &\quad -\frac{1}{2}(-4-6\,n+k-2\,n^2+18\,k^2+12\,k^3+30\,k^2\,n\\ &\quad +9\,k\,n+4\,n^3\,k+12\,n^2\,k+12\,k^2\,n^2\\ &\quad +20\,n\,k^3+8\,n^2\,k^3)(\,k+1\,)\,/(\,(\,n-k\,)\\ &\quad (\,2\,k+1\,)\,(\,n+2+k\,)\,(\,n-k+1\,)\,)\end{aligned}$$

This means in a more common notation

$$\begin{aligned}&2(n+1)(2n+3)F(n,k)\\ &-(4n+7)(2n+1)F(n+1,k)\\ &+(2n+3)(2n+1)F(n+2,k)\\ &=G(n,k+1)-G(n,k)\end{aligned} \tag{24}$$

with $G(n,k) := F(n,k)R(n,k)$ and

```
> R:=-1/2*(-4-6*n+k-2*n^2+12*n^2*k^2
> +12*n^2*k+30*k^2*n+9*k*n+12*k^3
> +18*k^2+8*n^2*k^3)*(k+1)/
> ((n-k)*((2*k+1)*(n+2-k)*(n-k+1)):
```

Summing this over $k \ge 1$ we find

$$\begin{aligned}&2(n+1)(2n+3)F(n)-(4n+7)(2n+1)F(n+1)\\ &+(2n+3)(2n+1)F(n+2)\\ &=\frac{(n+1)(4n^2+26n+27)}{2(n+2)(n+3)}\,.\end{aligned} \tag{25}$$

We only have to check whether

$$Ftest(n) := 1 + \frac{H_n}{2} - n + nH_{n+1} - \frac{4^n}{\binom{2n}{n}} \tag{26}$$

fulfills this recursion, which MAPLE confirms very easily:

```
> H:=n->sum(1/k,k=1..n):
> Ftest:=n->1+H(n)/2-n+n*H(n+1)
> -4^n/binomial(2*n,n):
> normal(expand(2*(2*n+3)*(n+1)*Ftest(n)
> -(4*n+7)*(2*n+1)*Ftest(n+1)
> +(2*n+3)*(2*n+1)*Ftest(n+2)-
> (n+1)*(4*n^2*26*n+27)/2/(n+2)/(n+3)));
```

$$0$$

Two initial conditions also match, and the proof is finished.

• SUM_3

For SUM_3 we do this. From experiments with GFUN, we 'know' the correct answer, and we only want to *prove* it with Zeilberger's algorithm.

Our general term is now this:

```
> F:=1/binomial(2*n+1,n)^2*(2*k+1)^2/
> k/(k+1)*binomial(2*n+1,n-k)^2:
> zeil(F,k,n);
```

$$\begin{aligned}&-(\,5+4\,n\,)\,(\,4\,n+3\,)\,(\,n+1\,)+2\,(\,2\,n+3\,)\,(\,2\,n+1\,)^2\,N,\\ &\quad -(\,k+1\,)(-10\,k^3-8\,n\,k^3+5\,k^2+4\,k^2\,n+18\,k\\ &\quad +16\,n^3\,k+54\,n^2\,k+56\,n\,k-9\\ &\quad -27\,n^2-28\,n-8\,n^3)(\,n+1\,)\\ &\quad /((\,n-k+1\,)^2\,(\,2\,k+1\,))\end{aligned}$$

```
> recu:="[1];
```

$$\begin{aligned}recu &:= -(\,5+4\,n\,)\,(\,4\,n+3\,)\,(\,n+1\,)\\ &\quad +2\,(\,2\,n+3\,)\,(\,2\,n+1\,)^2\,N\end{aligned}$$

This is the recursion of the $F(n,k)$; as above, it equals $G(n,k+1) - G(n,k)$. We sum it up on $k \ge 1$, and end up with $-G(n,1)$. The latter quantity gets the name 'rside'.

```
> rside:=-subs(k=1, F)*subs(k=1,""[2]);
```

$$\begin{aligned}rside &:= 3(\,8\,n^3+27\,n^2+24\,n+4\,)\,(\,n+1\,)\\ &\quad \mathrm{binomial}(\,2\,n+1,n-1\,)^2\Big/\\ &\quad (n^2\,\mathrm{binomial}(\,2\,n+1,n\,)^2\,)\end{aligned}$$

Now we check whether our guessed answer, which we call 'Ftest', fulfills the recursion.

```
> Ftest:=n->binomial(4*n+1,2*n)/
> binomial(2*n,n)^2-3+2/(n+1);
```

$$Ftest := n \to \frac{\mathrm{binomial}(\,4\,n+1,2\,n\,)}{\mathrm{binomial}(\,2\,n,n\,)^2} - 3 + 2\,\frac{1}{n+1}$$

```
> normal(expand(-(5+4*n)*(4*n+3)*(n+1)
> *Ftest(n)+2*(2*n+3)*(2*n+1)^2
> *Ftest(n+1)-rside));
```

$$0$$

This is the case, and since initial conditions also match, we are done.

• SUM_4

We ask MAPLE to perform

$$A := \sum_{k=0}^{n+1} \binom{2n+2}{n+1-k}^2 k^2 \tag{27}$$

by saying

```
> simplify(sum(binomial(2*n+2,n+1-k)^2
> *k^2,k=0..n+1));
```

with the result

$$2\frac{16^n(n+1)\Gamma(3/2+2n)}{\pi^{1/2}\Gamma(2n+2)},$$

which we rewrite by hand as

$$A = \frac{n+1}{2}\binom{4n+2}{2n+1}. \tag{28}$$

The other part,

$$B := \sum_{k=0}^{n+1} \binom{2n+2}{n+1-k}^2 \tag{29}$$

looks simpler, but is actually harder. However, we can help MAPLE by noticing a symmetry. We ask for the "full" sum

```
> simplify(sum(binomial(2*n+2,k)^2,
> k=0..2*n+2));
```

with the result

$$\frac{\Gamma(5+4n)}{\Gamma(2n+3)^2},$$

which is simply

$$\binom{4n+4}{2n+2}. \tag{30}$$

To this quantity we add the "middle" element $\binom{2n+2}{n+1}^2$ and obtain our sum as half of that;

$$B = \frac{1}{2}\left(\binom{4n+4}{2n+2} + \binom{2n+2}{n+1}^2\right). \tag{31}$$

Computing $(4n+1)A - n(n+1)B$ and subtracting the missing summands (for $k = 0$ and $k = n+1$), we get the result from Theorem 4.

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Also, I thank Arnold Knopfmacher for inviting me to South Africa, where I learnt about the papers [1, 2].

References

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