SOME IDENTITIES FOR CHEBYSHEV POLYNOMIALS

PETER J. GRABNER† AND HELMUT PRODINGER‡

Abstract. We prove a generalization of a conjectured formula of Melham and provide some background about the involved (Chebyshev) polynomials.

1. Introduction

In [3] Melham considered the two sequences

$$\begin{aligned} U_n &= pU_{n-1} - U_{n-2}, \qquad & U_0 = 0,\ U_1 = 1,\\ V_n &= pV_{n-1} - V_{n-2}, \qquad & V_0 = 2,\ V_1 = p, \end{aligned}$$

and conjectured the formula

$$U_n^{2k} + U_{n+1}^{2k} = \sum_{r=0}^{k} \frac{D^r V_k}{r!}\, U_n^{k-r} U_{n+1}^{k-r},$$

where D means differentiation with respect to p. We remark here that up to simple changes of variable these polynomials are Chebyshev polynomials. More precisely

$$\begin{aligned} U_n(p) &= \mathcal{U}_{n-1}(\tfrac{p}{2}),\\ V_n(p) &= 2\mathcal{T}_n(\tfrac{p}{2}), \end{aligned}$$

where $\mathcal{T}_n$ and $\mathcal{U}_n$ denote the classical Chebyshev polynomials of first and second kind, respectively.

The aim of this paper is to prove a general identity that contains Melham's conjecture as a special case: Set $W_n = aU_n + bV_n$ and $\Omega = a^2 + 4b^2 - b^2p^2$, then

$$W_n^{2k} + W_{n+1}^{2k} = \sum_{r=0}^{k} \Omega^{k-r}\, \lambda_{k,r}\, W_n^r W_{n+1}^r, \tag{1}$$

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with

$$\lambda_{k,r} = \sum_{0\le 2j\le r} (-1)^j \frac{k(k-1-j)!}{(k-r)!j!(r-2j)!} p^{r-2j}$$

and $\lambda_{0,0} = 2$.

From [1, 2] we know explicit expansions for Chebyshev polynomials:

$$V_j = \sum_{0\le 2k\le j} (-1)^k \binom{j-k}{k} \frac{j}{j-k}\, p^{j-2k}$$

for $j \ge 1$ and $V_0 = 2$. Then we have

$$\lambda_{k,r} = \frac{D^{k-r} V_k}{(k-r)!},$$

which links Melham's conjecture and (1).

2. Proof of the Formula

In order to prove (1) we form the generating function

$$g(z) = \sum_{k=0}^{\infty} z^k \sum_{r=0}^{k} \Omega^{k-r}\, \lambda_{k,r}\, \sigma^r$$

with $\sigma = W_n W_{n+1}$. We reorder this to obtain (setting $r - k = t$)

$$\begin{aligned}
g(z) &= \sum_{j\ge 1}\sum_{k\ge 2j}\sum_{t\ge 0} z^t \Omega^t (\sigma z)^k (-1)^j \frac{(k+t)(k+t-1-j)!}{t!j!(k-2j)!} p^{k-2j} \\
&\quad + \sum_{k\ge 0}\sum_{t\ge 0} z^t \Omega^t (\sigma z)^k \frac{(k+t)!}{t!k!} p^k + 1 \\
&= \sum_{j\ge 1}\sum_{k\ge 2j} (-1)^j p^{k-2j} (\sigma z)^k (1-\Omega z)^{j-k-1} \frac{(k-j-1)!}{j!(k-2j)!}(k - j\Omega z) \\
&\quad + \sum_{k\ge 0} (p\sigma z)^k (1-\Omega z)^{-k-1} + 1 \\
&= \sum_{j\ge 1} (-1)^j (\sigma z)^{2j} (1-\Omega z)^{-j-1} \bigl(2 - (\Omega + p\sigma) z\bigr) \\
&\quad + \frac{1}{1-(\Omega+p\sigma)z} + 1 \\
&= -\frac{(\sigma z)^2 \bigl(2-(\Omega+p\sigma)z\bigr)}{\bigl(1-(\Omega+p\sigma)z\bigr)\bigl(1-(\Omega+p\sigma)z+\sigma^2 z^2\bigr)} + \frac{1}{1-(\Omega+p\sigma)z} + 1 \\
&= \frac{2-(\Omega+p\sigma)z}{1-(\Omega+p\sigma)z+\sigma^2 z^2}.
\end{aligned}$$

The generating function of the left hand side is

$$\frac{1}{1-W_n^2 z}+\frac{1}{1-W_{n+1}^2 z}=\frac{2-(W_n^2+W_{n+1}^2)z}{1-(W_n^2+W_{n+1}^2)z+W_n^2W_{n+1}^2z^2}$$

and the assertion follows from

$$W_n^2+W_{n+1}^2=pW_nW_{n+1}+\Omega,$$

which is easily proved e. g. by using the explicit forms

$$U_n=\frac{\alpha^n-\beta^n}{\alpha-\beta},\qquad V_n=\alpha^n+\beta^n,$$

with

$$\alpha=\frac{p+\sqrt{p^2-4}}{2},\qquad \beta=\frac{p-\sqrt{p^2-4}}{2}.$$

3. Further identities

Many other similar formulæ seem to exist; we just give one other example; set

$$a_{k,r}=\sum_{0\le\lambda\le r}(-1)^\lambda p^{2k-2\lambda}\frac{k(k-\lfloor\frac{\lambda}{2}\rfloor-1)!2^{\lceil\frac{\lambda}{2}\rceil}}{(k-r)!\lambda!(r-\lambda)!}\prod_{i=0}^{\lfloor\frac{\lambda}{2}\rfloor-1}\Big(2k-2\Big\lceil\frac{\lambda}{2}\Big\rceil-1-2i\Big)$$

and $a_{0,0}=2$, then

$$W_n^{2k}+W_{n+2}^{2k}=\sum_{r=0}^{k}\Omega^{k-r}\,a_{k,r}\,W_n^rW_{n+2}^r.$$

The proof is as before.

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(P. Grabner) Institut für Mathematik A, Technische Universität Graz, Steyrergasse 30, 8010 Graz, Austria

E-mail address: grabner@weyl.math.tu-graz.ac.at

(H. Prodinger) The John Knopfmacher Centre for Applicable Analysis and Number Theory, Department of Mathematics, University of the Witwatersrand, P. O. Wits, 2050 Johannesburg, South Africa

E-mail address: helmut@gauss.cam.wits.ac.za