MORTENSON'S IDENTITIES AND PARTIAL FRACTION DECOMPOSITION

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Abstract. We reprove two identities of Mortenson by using not more than partial fraction decomposition.

1. Introduction

Mortenson [2, 3] proved the two identities

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}(H_{n+k}+H_{m+k}-2H_k)=0, \tag{1}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\Big(1+k(H_{n+k}+H_{m+k}-2H_k)\Big)=1+n+m. \tag{2}$$

Here, $1\le m\le n$ and $H_n=\sum_{1\le k\le n}\frac{1}{k}$ are harmonic numbers.

The goal of the present note is to show that the particularly simple technique, presented in [4], works here as well. Because of a formula, e. g., discussed in [4], identity (2) is equivalent to the more appealing

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}k(H_{n+k}+H_{m+k}-2H_k)=n+m.$$

Compare also with the paper [5].

The method consists of the following steps:

- A rational function, of the form
$$\frac{(z+1)\dots(z+n)}{z(z-1)\dots(z-n)}\cdot Y,$$
will undergo *partial fraction decomposition.*
- The resulting equation will be multiplied by z, and the limit $z\to\infty$ will be performed.
- We write harmonic numbers as
$$H_k=\sum_{j\ge1}\left[\frac{1}{j}-\frac{1}{j+k}\right]=\sum_{j\ge1}\frac{k}{j(j+k)}.$$

We use an individual term $\frac{z}{j(j+z)}$ as part of the rational function. The resulting identities will be summed over all $j \geq 1$.
- To obtain the final result, these infinite series have to be evaluated. We use *creative telescoping*, as popularized in the book [1] to do that.

Factorials are, whenever necessary, defined by the Gamma function. In particular, $1/(-n)!$, for $n \in \mathbb{N}$, must be interpreted, as usual, by 0.

2. The first identity

The two results

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} H_k = 2H_n,$$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} H_{n+k} = 2H_n,$$

appear already in [4], so let us concentrate on $(0 \leq m \leq n)$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} H_{m+k}.$$

Consider

$$A := \frac{(z+1)\dots(z+n)}{z(z-1)\dots(z-n)} \frac{m+z}{j(j+m+z)}.$$

Then partial fraction decomposition results in

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} \frac{m+k}{j(j+m+k)} \frac{1}{z-k}$$
$$= A - \frac{(j+m-1)!^2}{(j+m-n-1)!(j+m+n)!} \frac{1}{j+m+z}.$$

Multiplying this by z and then performing the limit $z \to \infty$ leads to

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} \frac{m+k}{j(j+m+k)}$$
$$= \frac{1}{j} - \frac{(j+m-1)!^2}{(j+m-n-1)!(j+m+n)!}.$$

Summing over $j \geq 1$, we get

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} H_{m+k} \\ = \sum_{j\geq 0} \left[\frac{1}{j+1} - \frac{(j+m)!^2}{(j+m-n)!(j+1+m+n)!} \right].$$

Denoting the term in the sum that still needs to be evaluated by $F(n,j)$, we find

$$F(n+1,j) - F(n,j) = G(n,j+1) - G(n,j),$$

with

$$G(n,j) = \frac{2}{n+1} \frac{(j+m)!^2}{(j+m-n-1)!(j+1+m+n)!}.$$

(Once this is known, it is trivial to check.)

We need the summatory functions

$$S(n) = \sum_{j\geq 0} F(n,j);$$

m is always treated as a parameter. We find by summing

$$S(n+1) - S(n) = \lim_{J\to\infty} \big(G(n,J) - G(n,0)\big) = \frac{2}{n+1}, \qquad n \geq m,$$

so that

$$S(n) - S(m) = 2(H_n - H_m).$$

The instance $S(m) = 2H_m$ is already known, as mentioned, and we get for $0 \leq m \leq n$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} H_{m+k} = 2H_n.$$

Putting the individual evaluations together, we get identity (1).

3. The second identity

Consider the partial fraction decomposition

$$\frac{(z+1)\dots(z+n)}{(z-1)\dots(z-n)} \frac{m+z}{j(j+m+z)} \\ = \sum_{k=1}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} \frac{k(m+k)}{j(j+m+k)} \frac{1}{z-k} \\ - \frac{(m+j-1)!(m+j)!}{(j+m-n-1)!(m+n+j)!} \frac{1}{j+m+z} + \frac{1}{j}.$$

Again, multiplying this by z and taking the limit $z \to \infty$ results in

$$\sum_{k=1}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} \frac{k(m+k)}{j(j+m+k)} = \frac{n(n+1)}{j+1} - 1 + \frac{(m+j)!(m+j+1)!}{(j+m-n)!(m+n+j+1)!},$$

which we sum:

$$\sum_{k=1}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} k H_{m+k} = \sum_{j \ge 0} \left[\frac{n(n+1)}{j+1} - 1 + \frac{(m+j)!(m+j+1)!}{(j+m-n)!(m+n+j+1)!} \right].$$

Denoting the term in the bracket of the right-hand side by $F(n,j)$, we get

$$nF(n+1,j) - (n+2)F(n,j) = G(n,j+1) - G(n,j) + 2,$$

with

$$G(n,j) = -2 \frac{(j+m)!(j+m+1)!}{(j+m-n-1)!(j+m+n+1)!},$$

which is again routine to check. Consequently we find for the summatory functions the recursion

$$nS(n+1) - (n+2)S(n) = \lim_{J \to \infty} [G(n,J) + 2J - G(n,0)] = 2n(n+2) - 2m.$$

The recursion is valid for $n \ge m$. However, the special case

$$S(m) = 2m(m+1)H_m - m^2$$

is already known, see [4], and therefore we get in general

$$S(n) = 2n(n+1)H_n - n^2 - n + m.$$

This is easy to check by induction.

So we have the general formula for $0 \le m \le n$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} k H_{m+k} = 2n(n+1)H_n - n^2 - n + m.$$

To evaluate Mortenson's second sum, we add the ingredients

$$2n(n+1)H_n - n^2 - n + m + 2n(n+1)H_n - n^2 - 4n(n+1)H_n + 2n(n+1) = m + n$$

and are done.

References

[1] Marko Petkovšek, Herbert S. Wilf, and Doron Zeilberger. *$A = B$*. A K Peters Ltd., Wellesley, MA, 1996. With a foreword by Donald E. Knuth, with a separately available computer disk.

[2] E. Mortenson. Supercongruences between truncated ${}_2F_1$ hypergeometric functions and their Gaussian analogs. *Trans. Amer. Math. Soc.*, 335 (2003), 139–147.

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[4] H. Prodinger. Human proofs of identities by Osburn and Schneider. *INTEGERS*, 8 (2008), paper A10 (8 pages).

[5] H. Prodinger. Identities involving harmonic numbers that are of interest for physicists. *Utilitas Mathematica*, 83 (2010), 291–299.

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