A note on a paper of G. H. Weiss and M. Dishon

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Yesterday[1] I browsed through old volumes of *The Fibonacci Quarterly* (which I can recommend to everyone) and found [2]: In an attempt to improve on earlier result of Narayana and Kreweras, the authors prove that for $r, s \geq 1$

$$[u^r v^s] \frac{1 - u - v - \sqrt{1 - 2(u+v) + (u-v)^2}}{2} = \frac{1}{r+s-1} \binom{r+s-1}{r} \binom{r+s-1}{s}$$

(the notation $[u^r v^s] f(u, v)$ refers to the coefficient of $u^r v^s$ in the power series expansion of $f(u, v)$).

However, the really easy method in this context is the *Lagrange inversion formula,* as I will sketch now. Let $S = \bigl(1 - u - v - \sqrt{1 - 2(u+v) + (u-v)^2}\bigr)/2$, then $S^2 + (u + v - 1)S + uv = 0$, or

$$u = \frac{S}{\Phi(S)} \quad \text{with} \quad \Phi(S) = \frac{v + S}{1 - v + S}.$$

Now the Lagrange inversion formula tells us (see, e. g., [1]) that

$$[u^r] S = \frac{1}{r} [S^{r-1}] \bigl(\Phi(S)\bigr)^r,$$

[1] October 16, 2001

or, with $v = St$,

$$
\begin{aligned}
[u^r v^s]S &= \frac{1}{r}[S^{r-1}v^s]\left(\frac{v+S}{1-v+S}\right)^r \\
&= \frac{1}{r}[S^{r+s-1}t^s]\left(\frac{S(1+t)}{1-S(1+t)}\right)^r \\
&= \frac{1}{r}[t^s](1+t)^r[S^{s-1}]\big(1-S(1+t)\big)^{-r} \\
&= \frac{1}{r}[t^s](1+t)^{r+s-1}\binom{r+s-2}{s-1} \\
&= \frac{1}{r}\binom{r+s-1}{s}\binom{r+s-2}{s-1},
\end{aligned}
$$

which is clearly equivalent to the statement to be proved.

References

1. R. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge University Press, Cambridge, 1999.
2. G. H. Weiss and M. Dishon, *A method for the evaluation of certain sums involving binomial coefficients*, Fibonacci Quarterly **14** (1976), 75–77.

A. M. S. Classification: 05A10