PERIODIC OSCILLATIONS IN THE ANALYSIS OF ALGORITHMS

HELMUT PRODINGER

Abstract. A large number of results in analysis of algorithms contain fluctuations. A typical result might read "The expected number of ... for large n behaves like $\log_2 n + \text{constant} + \delta(\log_2 n)$, where $\delta(x)$ is a periodic function of period one and mean zero." Examples include various trie parameters, approximate counting, probabilistic counting, radix exchange sort, leader election, skip lists, adaptive sampling. Often, there are huge oscillations to be noted, especially if one wants to compute variances. In order to see this, one needs identities for the Fourier coefficients of the periodic functions involved. There are several methods to derive such identities, which belong to the realm of modular functions. The most flexible one seems to be the calculus of residues. In some situations, Mellin transforms help. Often, known identities can be employed. This survey shows the various techniques by elaborating on the most important examples from the literature.

1. Introduction

A surprisingly large number of results in *analysis of algorithms* contain *fluctuations*. A typical result might read "The expected number of ... for large n behaves like $\log_2 n + \text{constant} + \delta(\log_2 n)$, where $\delta(x)$ is a periodic function of period one and mean zero." Examples include various trie parameters, approximate counting, probabilistic counting, radix exchange sort, leader election, skip lists, adaptive sampling; see the classic books by Flajolet, Knuth, Mahmoud, Sedgewick, Szpankowki [20, 14, 15, 16, 21] for background.

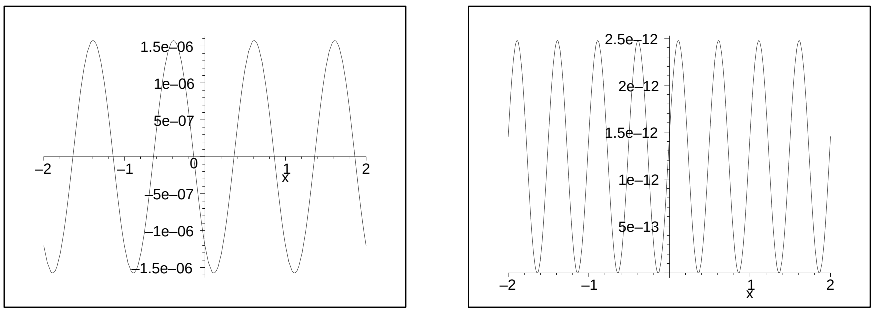

Figure 1. $\delta_0(x)$ and $\delta_0^2(x)$

As one can see from the picture, $\delta_0(x)$ has mean zero (the zeroth Fourier coefficient is not there). On the other hand, $\delta_0^2(x)$ is still periodic with period 1, but its mean is *not* zero. Why should we worry about a quantity apparently as small as $\approx 10^{-12}$?

Key words and phrases. Periodic oscillations, residues, Mellin transform, Dedekind's eta functions, modular functions, analysis of algorithms, tries, approximate counting, geometric random variables.

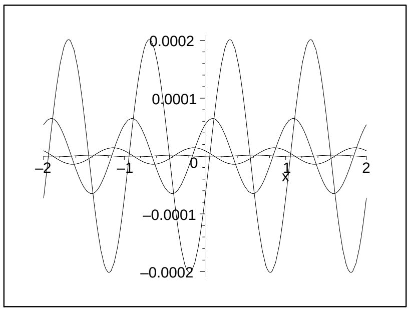

FIGURE 2. The functions $\frac{1}{\log 2}\sum_{k\neq 0}\frac{\Gamma(j-\frac{2\pi ik}{\log 2})}{j!}e^{2\pi ikx}$ grow in amplitude with j

The reason is the *variance* of such parameters, as it naturally contains the term "$-$expectation^2," and as such also $-\delta^2(x)$. That might not be a sufficient motivation for a casual reader if it were not the case that often *substantial cancellations* occur. In order to identify them, one has to know more about $\delta^2(x)$. If one ignores these terms, one gets wrong results, and the results are not wrong by $\approx 10^{-12}$, but *by an order of growth!* Path length in tries, Patricia tries, and digital search trees [7, 13, 9] are such cases: the variance is in reality of order n only, but ignoring the fluctuations would lead to a (wrong) $\approx n^2$ result.

Questions like that occurred in several writings of this author (together with various coauthors), as can be seen from the references. The techniques are extremely interesting, as one has to dig deep into classical analysis. So far, it seems that the *calculus of residues* is the most versatile approach in this context. Another approach is to use (modular) identities due to Dedekind, Ramanujan, Jacobi and others (which can often be proved by Mellin transform techniques); however, often they do not quite *fit*. The residue calculus approach directly addresses the formula that is ultimately needed.

In this survey paper, we discuss all these methods by looking at various examples.

Oscillating functions are usually given as *Fourier* series $f = \sum_{k\neq 0} a_k e^{2\pi ikx}$, thus representing a periodic function of period 1, and since the term a_0 is missing, oscillating around zero. We often refer to the coefficient a_k by writing $[f]_k$.

Here are some examples from the literature.

Approximate counting. [5, 10, 18, 19]

After n successive increments the average content $\overline{C}_n$ of the counter satisfies:

$$\overline{C}_n \sim \log_2 n + \frac{\gamma}{L} - \alpha + \frac{1}{2} - \delta_0(\log_2 n),$$

with

$$\alpha = \sum_{k\geq 1}\frac{1}{2^k-1} \quad \text{and} \quad \delta_0(x) = \frac{1}{L}\sum_{k\neq 0}\Gamma(-\chi_k)e^{2\pi ikx},$$

with $L = \log 2$ and $\chi_k = \frac{2\pi ik}{L}$. The identity that one needs is

$$[\delta_0^2]_0 = \frac{1}{L^2}\sum_{k\neq 0}\Gamma(\chi_k)\Gamma(-\chi_k) = \frac{\pi^2}{6L^2} - \frac{11}{12} - \frac{2}{L}\sum_{h\geq 1}\frac{(-1)^{h-1}}{h\,(2^h-1)}. \tag{1.1}$$

Maximum of a sample of n geometric random variables. [22, 11]

Assume that X is a geometric random variable such that $\mathbb{P}\{X = k\} = 2^{-k}$ (for simplicity, we only discuss this case, not the slightly more general $\mathbb{P}\{X = k\} = (1 - q)q^{k-1}$). We consider n independent trials and look for the maximum of them. This is a natural parameter which is also useful in the analysis of various algorithms (e.g., skiplists [12]).

The expected value is given by

$$E_n \sim \log_2 n + \frac{\gamma}{L} + \frac{1}{2} - \delta_0 (\log_2 n)$$

with the same periodic function as before.

Tries. [10, 7]

The expected number of internal nodes in a trie built from n random data

$$l_n \sim \frac{n}{L} + n\sigma(\log_2 n),$$

with

$$\sigma(x) = \frac{1}{L} \sum_{k \neq 0} \chi_k \Gamma(1 - \chi_k) e^{2\pi i k x}.$$

The formula that one needs is

$$[\sigma^2]_0 = 3 - \frac{1}{L} - \frac{1}{L^2} + \frac{2}{L} \sum_{j \geq 2} \frac{(-1)^j j}{(j+1)(j-1)(2^j - 1)}.$$

Partial match queries in tries. [8]

The average cost (defined in the paper [8]), for random tries constructed from n random data, is

$$l_n \sim \sqrt{n} \Big(\sqrt{\pi} \frac{1 + \sqrt{2}}{2L} + \tau \big(\log_2 \sqrt{n} \big) \Big),$$

where the fluctuating function $\tau(x) = \sum_{k \neq 0} \tau_k e^{2k\pi i x}$ has the Fourier coefficients

$$\tau_k = \frac{1}{2L} \Big(1 + \sqrt{2} (-1)^k \Big) \Gamma \Big(\frac{-1 - \chi_k}{2} \Big) \Big(\frac{-1 + \chi_k}{2} \Big). \tag{1.2}$$

The formula one needs is

$$[\tau^2]_0 = \frac{3}{4L} - \frac{\pi}{4L^2} (3 + 2\sqrt{2}) + \frac{3 - 2\sqrt{2}}{L} F(L) + \frac{2\sqrt{2}}{L} F\Big(\frac{L}{2}\Big)$$

with

$$F(x) = \sum_{k \geq 1} \frac{e^{-kx}}{1 + e^{-2kx}}.$$

2. Proofs by residue calculus

The following approach to evaluate $[\delta^2]_0$ seems to be the easiest and most flexible. We start with the following example:

$$\delta_0(x) = \frac{1}{L}\sum_{k\neq 0}\Gamma(-\chi_k)e^{2\pi ikx}.$$

Find a function $F(z)$ so that $[\delta_0^2]_0$ is (apart from a few extra terms) the sum of the residues along the imaginary axis. Here, take

$$F(z) = \frac{L}{e^{Lz}-1}\Gamma(-z)\Gamma(z).$$

If we set

$$I_1 = \frac{1}{2\pi i}\int_{\frac12 - i\infty}^{\frac12 + i\infty} F(z)dz,$$

then by shifting and collecting residues,

$$I_1 = \frac{1}{2\pi i}\int_{-\frac12 - i\infty}^{-\frac12 + i\infty} F(z)dz + \sum_{k\neq 0}\Gamma(-\chi_k)\Gamma(\chi_k) - \frac{\pi^2}{6} - \frac{L^2}{12}.$$

Now one writes

$$\frac{1}{e^z-1} = -1 - \frac{1}{e^{-z}-1}$$

and gets, by a simple change of variable $z := -z$,

$$I_1 = -\frac{1}{2\pi i}\int_{-\frac12 - i\infty}^{-\frac12 + i\infty}\Gamma(-z)\Gamma(z)dz - I_1 + \sum_{k\neq 0}\Gamma(-\chi_k)\Gamma(\chi_k) - \frac{\pi^2}{6} - \frac{L^2}{12}.$$

The integral

$$I_2 = -\frac{1}{2\pi i}\int_{-\frac12 - i\infty}^{-\frac12 + i\infty}\Gamma(-z)\Gamma(z)dz$$

can be computed by collecting the negative residues right to the line $\Re z = -\frac12$, viz.

$$I_2 = -\frac{1}{2\pi i}\int_{-\frac12 - i\infty}^{-\frac12 + i\infty}\Gamma(-z)\Gamma(z)dz = \sum_{l\ge 1}\frac{(-1)^l}{l!}(l-1)! = -L.$$

Altogether we have

$$2I_1 = -L + \sum_{k\neq 0}\Gamma(-\chi_k)\Gamma(\chi_k) - \frac{\pi^2}{6} - \frac{L^2}{12}.$$

On the other hand, integral I_1 is also the sum of the negative residues right of the line $\Re z = \frac12$, i. e.,

$$I_1 = -L\sum_{l\ge 1}\frac{(-1)^l}{l!(2^l-1)}(l-1)! = -L\sum_{l\ge 1}\frac{(-1)^l}{l(2^l-1)}.$$

Combining these results, we get

$$-2L\sum_{l\ge 1}\frac{(-1)^l}{l(2^l-1)} = -L + \sum_{k\neq 0}\Gamma(-\chi_k)\Gamma(\chi_k) - \frac{\pi^2}{6} - \frac{L^2}{12}.$$

This is the identity we wanted.

With not much more effort one can also compute the coefficients $[\delta_0^2]_k$, for $k \neq 0$. For this, one works with the function

$$F(z) = \frac{L}{e^{Lz}-1}\Gamma(-z-\chi_k)\Gamma(z).$$

One obtains

$$\begin{aligned}
[\delta_0^2]_k &= \frac{1}{L^2}\sum_{j\neq 0,\,\neq k}\Gamma(-\chi_j)\Gamma(-\chi_k+\chi_j)\\
&= \frac{2}{L}\sum_{l\ge 1}\frac{(-1)^l\Gamma(-\chi_k+l)}{l!(2^l-1)} + \frac{2}{L^2}\Gamma(-\chi_k)\big(\psi(-\chi_k)+\gamma\big).
\end{aligned}$$

We omit the details.

Guy Louchard, who is interested in higher moments, asked to compute the coefficients $[\delta_0^3]_k$. Here is the instance $k=0$, the general case is very involved and not too attractive:

$$\begin{aligned}
[\delta_0^3]_0 = -1 &- \frac{2\zeta(3)}{L^3} - \frac{1}{L}\sum_{l\ge 1}\frac{(-1)^l}{l(2^l-1)} + \frac{6}{L^2}\sum_{l\ge 1}\frac{(-1)^l H_{l-1}}{l(2^l-1)} + \frac{2\log 3}{L}\\
&+ \frac{2}{L}\sum_{l,j\ge 1}\frac{(-1)^{l+j}}{(l+j)(2^l-1)}\left[\frac{1}{2^j-1}+\frac{1}{2^{j+l}-1}\right]\binom{l+j}{j}.
\end{aligned}$$

(In this formula, the harmonic numbers $H_n := \sum_{1\le k\le n}\frac{1}{k}$ appear.)

This has been tested numerically as well and gives $9.4281776309579660642190 3\times 10^{-25}$.

Let us straight ahead do another example, which also occurs often:

$$\sigma(x) = \frac{1}{L}\sum_{k\neq 0}\chi_k\Gamma(-1-\chi_k)e^{2\pi i k x}.$$

Here, we take

$$F(z) = -\frac{L}{e^{Lz}-1}z^2\Gamma(-1-z)\Gamma(-1+z).$$

Then

$$I_1 = \frac{1}{2\pi i}\int_{-\frac12-i\infty}^{-\frac12+i\infty}F(z)dz + \sum_{k\neq 0}\chi_k(-\chi_k)\Gamma(-1-\chi_k)\Gamma(-1+\chi_k)+1$$

and

$$2I_1 = LI_2 + \sum_{k\neq 0}\chi_k(-\chi_k)\Gamma(-1-\chi_k)\Gamma(-1+\chi_k)+1$$

with

$$\begin{aligned}
I_2 &= \frac{1}{2\pi i}\int_{-\frac12-i\infty}^{-\frac12+i\infty}z^2\Gamma(-1-z)\Gamma(-1+z)dz = \sum_{l\ge 2}l^2\frac{(-1)^{l+1}}{(l+1)!}(l-2)! + \frac{L}{4}\\
&= \sum_{l\ge 2}\frac{(-1)^{l+1}l}{(l+1)(l-1)} = -L + \frac14 + \frac14 = -L + \frac12.
\end{aligned}$$

Therefore

$$2I_1 = -L^2 + \frac{L}{2} + \sum_{k\neq 0} \chi_k(-\chi_k)\Gamma(-1-\chi_k)\Gamma(-1+\chi_k) + 1.$$

But I_1 is also

$$I_1 = -\frac{L}{4} + L^2 + L\sum_{l\geq 2} \frac{l^2}{2^l-1}\frac{(-1)^{l+1}}{(l+1)!}(l-2)! = -\frac{L}{4} + L^2 + L\sum_{l\geq 2}\frac{(-1)^{l+1}l}{(2^l-1)(l+1)(l-1)}.$$

Putting things together, we find

$$\begin{aligned} 2I_1 &= -L^2 + \frac{L}{2} + \sum_{k\neq 0} \chi_k(-\chi_k)\Gamma(-1-\chi_k)\Gamma(-1+\chi_k) \\ &= -\frac{L}{2} + 2L^2 + 2L\sum_{l\geq 2}\frac{(-1)^{l+1}l}{(2^l-1)(l+1)(l-1)} + 1 \end{aligned}$$

or

$$\sum_{k\neq 0} \chi_k(-\chi_k)\Gamma(-1-\chi_k)\Gamma(-1+\chi_k) = -1 - L + 3L^2 + 2L\sum_{l\geq 2}\frac{(-1)^{l+1}l}{(2^l-1)(l+1)(l-1)},$$

which is the identity in question, as it expresses the quantity $L^2[\sigma^2]_0$ in two different ways.

Here is a third example, dealing with the function

$$\frac{1}{L}\sum_{k\geq 0}\Gamma(j-\chi_k)e^{2\pi ikx}$$

for $j\geq 1$ and the computation of the constant term of its square. The technique should be familiar by now. Consider the function

$$L\frac{\Gamma(j+z)\Gamma(j-z)}{e^{Lz}-1}.$$

Therefore we have

$$\begin{aligned} \sum_{k\neq 0}\Gamma(j+\chi_k)\Gamma(j-\chi_k) &= \frac{L}{2\pi i}\int_{\frac12 - i\infty}^{\frac12 + i\infty}\frac{\Gamma(j+z)\Gamma(j-z)}{e^{Lz}-1}dz \\ &\quad - \frac{L}{2\pi i}\int_{-\frac12 - i\infty}^{-\frac12 + i\infty}\frac{\Gamma(j+z)\Gamma(j-z)}{e^{Lz}-1}dz - \Gamma(j)^2. \end{aligned}$$

($\Gamma(j)^2$ is the residue at $z=0$.)

Now we use again the decomposition

$$\frac{1}{e^{Lz}-1} = -1 - \frac{1}{e^{-Lz}-1}$$

for the second integral and get

$$\begin{aligned}
&-\frac{L}{2\pi i}\int_{-\frac12-i\infty}^{-\frac12+i\infty}\frac{\Gamma(j+z)\Gamma(j-z)}{e^{Lz}-1}dz\\
&\quad=\frac{L}{2\pi i}\int_{-\frac12-i\infty}^{-\frac12+i\infty}\Gamma(j+z)\Gamma(j-z)dz+\frac{L}{2\pi i}\int_{-\frac12-i\infty}^{-\frac12+i\infty}\frac{\Gamma(j+z)\Gamma(j-z)}{e^{-Lz}-1}dz\\
&\quad=\frac{L}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(j+z)\Gamma(j-z)dz+\frac{L}{2\pi i}\int_{\frac12-i\infty}^{\frac12+i\infty}\frac{\Gamma(j-z)\Gamma(j+z)}{e^{Lz}-1}dz.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{k\neq 0}\left|\Gamma(j+\chi_k)\right|^2\\
&\quad=\frac{2L}{2\pi i}\int_{\frac12-i\infty}^{\frac12+i\infty}\frac{\Gamma(j+z)\Gamma(j-z)}{e^{Lz}-1}dz+\frac{L}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(j+z)\Gamma(j-z)dz-\Gamma(j)^2\\
&\quad=I_1+I_2-\Gamma(j)^2.
\end{aligned}$$

Integral I_1 is evaluated by shifting the contour to the *right* and collecting the *negative* residues, which gives

$$I_1=-2L\sum_{m\geq j}\frac{\Gamma(j+m)}{e^{Lm}-1}\frac{(-1)^{j-m+1}}{(m-j)!}$$

and with $m=h+j$

$$\begin{aligned}
&=2L\sum_{h\geq 0}\frac{(h+2j-1)!(-1)^h}{h!}\frac{1}{2^{h+j}-1}\\
&=2L(2j-1)!\sum_{h\geq 0}\binom{-2j}{h}\frac{1}{2^{h+j}-1}.
\end{aligned}$$

Integral I_2 is of interest for itself and appears already in early references to the *Mellin transform technique* as by Nielsen [17, p. 224]. (It could, however, by computed as in the previous examples.)

We start with the function

$$f(x)=\frac{x^j}{(1+x)^{2j}}$$

and perform its Mellin transform (see, e.g., [6] for definitions)

$$f^*(s)=\int_0^\infty f(x)x^{s-1}dx=B(j+s,j-s)=\frac{\Gamma(j+s)\Gamma(j-s)}{\Gamma(2j)}$$

with the Beta function $B(z,w)$ (compare [1]). The *fundamental strip* is $\langle -j,j\rangle$. Therefore the inversion formula for the Mellin transform gives

$$f(x)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(j+s)\Gamma(j-s)}{\Gamma(2j)}x^{-s}ds.$$

Now we may evaluate at $x=1$ and get the formula

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(j+s)\Gamma(j-s)ds=\Gamma(2j)2^{-2j}.$$

This produces the formula

$$\sum_{k\neq0}\Gamma(j+\chi_k)\Gamma(j-\chi_k)$$
$$=2L(2j-1)!\sum_{h\geq0}\binom{-2j}{h}\frac{1}{2^{h+j}-1}+L(2j-1)!2^{-2j}-(j-1)!^2.$$

Remark. The computation of the integral I_2 (as in the examples above) sometimes leads to series like

$$\sum_{l\geq1}(-1)^l l.$$

There is nothing wrong here. The correct interpretation is as an *Abel limit*

$$\lim_{t\to1-}\sum_{l\geq1}(-1)^l l t^l=\lim_{t\to1-}\frac{-t}{(1+t)^2}=-\frac{1}{4}.$$

3. Using the Mellin transform

Let us start with our running example (1.1) and show how this can be proved using the Mellin transform. This technique is very prominent in the analysis of algorithms, and we refer to [6] for a nice survey.

We might for instance start with the series

$$\sum_{h\geq1}\frac{(-1)^{h-1}}{h\,(2^h-1)}$$

and interpret it as $g(\log 2)$ with

$$g(x):=\sum_{h\geq1}\frac{(-1)^{h-1}}{h\,(e^{hx}-1)}=\sum_{h,k\geq1}\frac{(-1)^{h-1}}{h}e^{-hkx}.$$

Now one computes the Mellin transform $g^*(s)$:

$$g^*(s)=\sum_{h,k\geq1}\frac{(-1)^{h-1}}{h}e^{-hkx}=\sum_{h,k\geq1}\frac{(-1)^{h-1}}{h}h^{-s}k^{-s}\Gamma(s)=(1-2^{-s})\zeta(s+1)\zeta(s)\Gamma(s).$$

Using the inversion formula for the Mellin transform, one gets

$$\begin{aligned}g(x)&=\frac{1}{2\pi i}\int_{\frac32-i\infty}^{\frac32+i\infty}(1-2^{-s})\zeta(s+1)\zeta(s)\Gamma(s)x^{-s}ds\\&=\frac{\pi^2}{12x}-\frac{L}{2}+\frac{x}{24}+\frac{1}{2\pi i}\int_{-\frac32-i\infty}^{-\frac32+i\infty}(1-2^{-s})\zeta(s+1)\zeta(s)\Gamma(s)x^{-s}ds\\&=\frac{\pi^2}{12x}-\frac{L}{2}+\frac{x}{24}+\frac{1}{2\pi i}\int_{-\frac32-i\infty}^{-\frac32+i\infty}(2^s-1)\zeta(s+1)\zeta(s)\frac{1}{2\sqrt{\pi}}\Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{s+1}{2}\Big)x^{-s}ds.\end{aligned}$$

This form was obtained by taking 3 residues out and invoking the duplication formula of the Γ-function. (Observe that the exponential smallness of the Γ-function along vertical lines justifies the shifting of the line integral.) We now use the functional equation for $\zeta(s)$, namely

$$\Gamma\Big(\frac{s}{2}\Big)\zeta(s)=\pi^{s-\frac12}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s), \tag{3.1}$$

and continue:

$$\begin{aligned}
g(x)&=\frac{\pi^2}{12x}-\frac{L}{2}+\frac{x}{24}\\
&+\frac{1}{2\pi i}\int_{-\frac32-i\infty}^{-\frac32+i\infty}(2^s-1)\frac12\pi^{2s-\frac12}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s)\Gamma\Big(\frac{-s}{2}\Big)\zeta(-s)x^{-s}ds\\
&=\frac{\pi^2}{12x}-\frac{L}{2}+\frac{x}{24}\\
&+\frac{1}{2\pi i}\int_{\frac32-i\infty}^{\frac32+i\infty}(2^{-s}-1)\frac12\pi^{-2s-\frac12}\Gamma\Big(\frac{1+s}{2}\Big)\zeta(1+s)\Gamma\Big(\frac{s}{2}\Big)\zeta(s)x^{s}ds\\
&=\frac{\pi^2}{12x}-\frac{L}{2}+\frac{x}{24}-\frac{1}{2\pi i}\int_{\frac32-i\infty}^{\frac32+i\infty}(1-2^{-s})\pi^{-2s}\zeta(1+s)\zeta(s)\Gamma(s)x^s2^{-s}ds,
\end{aligned}$$

and so

$$g(x)=\frac{\pi^2}{12x}-\frac{L}{2}+\frac{x}{24}-g\Big(\frac{2\pi^2}{x}\Big). \tag{3.2}$$

This is the formula we need, since we can also rewrite the left side of (1.1) in terms of this $g(x)$ function:

$$[\delta_0^2]_0=\frac{1}{L^2}\sum_{k\neq0}\Gamma(\chi_k)\Gamma(-\chi_k)=\frac1L\sum_{k\ge1}\frac{1}{k\sinh(2k\pi^2/L)}=\frac2L\sum_{k\ge1}\frac{e^{kz}}{k(e^{2kz}-1)}$$

with $z=2\pi^2/L$. But

$$\begin{aligned}
\sum_{k\ge1}\frac{e^{kz}}{k(e^{2kz}-1)}&=\sum_{k\ge1,\,j\ge0}\frac1k e^{-k(2j+1)z}=\sum_{k\ge1,\,j\ge1}\frac1k e^{-kjz}-2\sum_{k\ge1,\,j\ge1}\frac{1}{2k}e^{-2kjz}\\
&=\sum_{k\ge1,\,j\ge1}\frac{(-1)^{k-1}}{k}e^{-kjz}=\sum_{k\ge1}\frac{(-1)^{k-1}}{k(e^{kz}-1)}=g(z),
\end{aligned}$$

and so

$$[\delta_0^2]_0=\frac2L g\Big(\frac{2\pi^2}{L}\Big).$$

Let us do a more complicated example in the same style: We want to rewrite $[\tau^2]_0$, with the Fourier coefficients given in (1.2). Note that

$$[\tau^2]_0=2\sum_{k\ge1}\tau_k\tau_{-k}=\frac{2}{4L^2}\sum_{k\ge1}\Big(3+2\sqrt2(-1)^k\Big)\Gamma\Big(\frac{1-\chi_k}{2}\Big)\Gamma\Big(\frac{1+\chi_k}{2}\Big).$$

Now we use the formula (reflection formula for the Gamma function, cf. [1]) $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$ and obtain

$$\Gamma\Big(\frac{1-\chi_k}{2}\Big)\Gamma\Big(\frac{1+\chi_k}{2}\Big) = \frac{\pi}{\sin(\pi/2 + ik\pi^2/L)} = \frac{\pi}{\cos(ik\pi^2/L)}$$
$$= \frac{\pi}{\cosh(k\pi^2/L)} = 2\pi\frac{e^{-k\pi^2/L}}{1+e^{-2k\pi^2/L}},$$

so that

$$[\tau^2]_0 = \frac{\pi}{L^2}\sum_{k\ge1}\Big(3+2\sqrt{2}(-1)^k\Big)\frac{e^{-k\pi^2/L}}{1+e^{-2k\pi^2/L}}. \tag{3.3}$$

Let us define two new functions

$$F(x) = \sum_{k\ge1}\frac{e^{-kx}}{1+e^{-2kx}} \qquad \text{and} \qquad G(x) = \sum_{k\ge1}\frac{(-1)^{k-1}e^{-kx}}{1+e^{-2kx}}.$$

Then, (3.3) in terms of $F(x)$ and $G(x)$ becomes

$$[\tau^2]_0 = \frac{3\pi}{L^2}F\Big(\frac{\pi^2}{L}\Big) - \frac{2\sqrt{2}\pi}{L^2}G\Big(\frac{\pi^2}{L}\Big). \tag{3.4}$$

We use a series transformation for $F(x)$ and $G(x)$. We start with

$$F(x) = \sum_{j\ge0}(-1)^j\sum_{k\ge1}e^{-k(2j+1)x} = \sum_{j\ge0}\chi(j)\frac{1}{e^{jx}-1}$$

where

$$\chi(j) = \begin{cases} 0 & \text{for } j \text{ even,}\\ 1 & \text{for } j \equiv 1 \bmod 4,\\ -1 & \text{for } j \equiv 3 \bmod 4.\end{cases}$$

Once we know

$$F(x) = \frac{\pi}{4x} - \frac{1}{4} + \frac{\pi}{x}F\Big(\frac{\pi^2}{x}\Big) \tag{3.5}$$

for $x > 0$, as we shall show soon, then $G(x) = F(x) - 2F(2x)$, hence

$$G(x) = \frac{1}{4} + \frac{\pi}{x}F\Big(\frac{\pi^2}{x}\Big) - \frac{\pi}{x}F\Big(\frac{\pi^2}{2x}\Big).$$

Applying the above to (3.4) we finally obtain

$$[\tau^2]_0 = \frac{3}{4L} - \frac{\pi}{4L^2}\big(3+2\sqrt{2}\,\big) + \frac{3-2\sqrt{2}}{L}F(L) + \frac{2\sqrt{2}}{L}F\Big(\frac{L}{2}\Big).$$

To prove (3.5) we proceed as follows. Let

$$\beta(s) = \sum_{j\ge0}(-1)^j\frac{1}{(2j+1)^s}.$$

We have

$$F(x) = \sum_{k\ge1}\frac{e^{-kx}}{1+e^{-2kx}} = \sum_{j\ge0}(-1)^j\sum_{k\ge1}e^{-k(2j+1)x},$$

so that the Mellin transform $F^*(s) = \int_0^\infty F(x)x^{s-1}dx$ of $F(x)$ becomes $F^*(s) = \Gamma(s)\zeta(s)\beta(s)$. By the Mellin inversion formula this yields

$$F(x) = \frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \Gamma(s)\zeta(s)\beta(s)x^{-s}ds.$$

Now we take the two residues $s = 1$ and $s = 0$ out from the above integral (observe that $\beta(0) = 1/2$ and $\beta(1) = \pi/4$, cf. [1]) and apply the duplication formula for $\Gamma(s)$ to obtain

$$F(x) = \frac{\pi}{4x} - \frac{1}{4} + \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{1}{\sqrt{\pi}}2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\Big(\frac{s+1}{2}\Big)x^{-s}\zeta(s)\beta(s)ds.$$

We now use the functional equations for $\zeta(s)$ and $\beta(s)$, namely

$$\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \pi^{s-\frac{1}{2}}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s)$$

and

$$\beta(1-s)\Gamma\Big(1-\frac{s}{2}\Big) = 2^{2s-1}\pi^{-s+\frac{1}{2}}\Gamma\Big(\frac{s+1}{2}\Big)\beta(s).$$

The first identity is Riemann's functional equation for $\zeta(s)$, and the second an immediate consequence of the functional equation for Hurwitz's ζ-function $\zeta(s,a)$ (cf. [2]), and the fact that

$$\beta(s) = 4^{-s}\Big[\zeta\big(s,\tfrac{1}{4}\big) - \zeta\big(s,\tfrac{3}{4}\big)\Big].$$

Substituting $1 - s = u$, we get

$$F(x) = \frac{\pi}{4x} - \frac{1}{4} + \frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \pi^{1-2u}\Gamma(u)x^{u-1}\zeta(u)\beta(u)du,$$

which proves (3.5).

Using the above scheme, several other identities which one needs in the analysis of algorithms can be proved. We refer to Szpankowski's book [21].

4. Modular identities

Formulæ like (3.2) belong to the realm of modular functions. Many of them can be found in the literature, and are due to Jacobi, Dedekind, Ramanujan and others. Berndt's book [4] contains a wealth of information about the subject, compare also [3].

Here is a little bit of background: Let H be the upper complex halfplane $\{z \in \mathbb{C} \mid \Im z > 0\}$. Then the Dedekind η function is defined by

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n\geq 1}\big(1 - e^{2\pi i n\tau}\big), \quad \tau \in H;$$

there is a transformation formula:

$$\eta\Big(-\frac{1}{\tau}\Big) = (-i\tau)^{1/2}\eta(\tau).$$

Ramanujan considered series

$$f(z) := \sum_{k\geq 1}\frac{k^m}{e^{2kz}-1}, \quad m \text{ an odd integer},$$

and could relate them to $f(\pi^2/z)$. The instance $m = -1$ is equivalent to the functional equation for Dedekind's eta function.

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*. Dover, 1973. A reprint of the tenth National Bureau of Standards edition, 1964.

[2] T. Apostol. *Introduction to Analytical Number Theory*. Springer, 1976.

[3] T. Apostol. *Modular Functions and Dirichlet Series in Number Theory*. Springer, 1978.

[4] B. Berndt. *Ramanujan's Notebooks. Part II*. Springer, 1989.

[5] P. Flajolet. Approximate counting: A detailed analysis. *BIT*, 25:113–134, 1985.

[6] P. Flajolet, X. Gourdon, and P. Dumas. Mellin transforms and asymptotics: Harmonic sums. *Theoretical Computer Science*, 144:3–58, 1995.

[7] P. Kirschenhofer, H. Prodinger, and W. Szpankowski. On the variance of the external path length in a symmetric digital trie. *Discrete Applied Mathematics*, 25:129–143, 1989.

[8] P. Kirschenhofer, H. Prodinger, and W. Szpankowski. Multidimensional digital searching and some new parameters in tries. *International Journal of Foundations of Computer Science*, 4:69–84, 1993.

[9] P. Kirschenhofer, H. Prodinger, and W. Szpankowski. Digital search trees again revisited: The internal path length perspective. *SIAM Journal on Computing*, 23:598–616, 1994.

[10] P. Kirschenhofer and H. Prodinger. On some applications of formulæ of Ramanujan in the analysis of algorithms. *Mathematika*, 38:14–33, 1991.

[11] P. Kirschenhofer and H. Prodinger. A result in order statistics related to probabilistic counting. *Computing*, 51:15–27, 1993.

[12] P. Kirschenhofer and H. Prodinger. The path length of random skip lists. *Acta Informatica*, 31:775–792, 1994.

[13] P. Kirschenhofer, H. Prodinger, and W. Szpankowski. On the balance property of Patricia tries: external path length viewpoint. *Theoret. Comput. Sci.*, 68:1–17, 1989.

[14] D. E. Knuth. *The Art of Computer Programming*, volume 1: Fundamental Algorithms. Addison-Wesley, 1973. Third edition, 1997.

[15] D. E. Knuth. *The Art of Computer Programming*, volume 3: Sorting and Searching. Addison-Wesley, 1973. Second edition, 1998.

[16] H. M. Mahmoud. *Evolution of Random Search Trees*. John Wiley, New York, 1992.

[17] N. Nielsen. *Handbuch der Theorie der Gammafunktion*. Teubner, 1906.

[18] H. Prodinger. Hypothetic analyses: Approximate counting in the style of Knuth, path length in the style of Flajolet. *Theoretical Computer Science*, 100:243–251, 1992.

[19] H. Prodinger. Approximate counting via Euler transform. *Mathematica Slovaka*, 44:569–574, 1994.

[20] R. Sedgewick and P. Flajolet. *An Introduction to the Analysis of Algorithms*. Addison-Wesley, 1996.

[21] W. Szpankowski. *Average case analysis of algorithms on sequences*. Wiley-Interscience, New York, 2001.

[22] W. Szpankowski and V. Rego. Yet another application of a binomial recurrence. Order statistics. *Computing*, 43:401–410, 1990.

Helmut Prodinger, The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits, 2050 Johannesburg, South Africa

E-mail address: `helmut@maths.wits.ac.za`