ON SOME CONTINUED FRACTION EXPANSIONS OF THE ROGERS-RAMANUJAN TYPE

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Abstract. By guessing the relative quantities and proving the recursive relation, we present some continued fraction expansions of the Rogers-Ramanujan type. Meanwhile, we also give some J-fraction expansions for the q-tangent and q-cotangent functions.

1. Introduction

In [17], the second author studied the functions

$$F(z) = \sum_{n\geq 0} \frac{(-1)^n z^n}{[2n+1]_q!} q^{dn^2},$$
$$G(z) = \sum_{n\geq 0} \frac{(-1)^n z^n}{[2n]_q!} q^{dn^2},$$

where

$$[n]_q := \frac{1-q^n}{1-q}, \qquad [n]_q! := [1]_q[2]_q \dots [n]_q,$$

and gave some continued fraction expansions of the q-tangent and q-cotangent functions for $d = 0, 1, 2$ in the following forms:

$$\frac{zF(z)}{G(z)} = \cfrac{z}{a_0 + \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{\ddots}}}}, \tag{1.1}$$

Date: April 12, 2010.

1991 *Mathematics Subject Classification.* 33D15, 11A55.

Key words and phrases. continued fraction, Rogers-Ramanujan type, q-tangent function, q-cotangent function.

[†] The first author was supported by the PCSIRT Project of the Ministry of Education, and the Specialized Research Fund for the Doctoral Program of Higher Education of China (200800551042).

[*] The second author was supported by International Science and Technology Agreement (Grant 67215).

$$\frac{zG(z)}{F(z)} = \cfrac{z}{a_0 + \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{\ddots}}}}, \tag{1.2}$$

which are variants of Jackson's [10]. Some special cases were considered in [5, 15, 16].

In [17], the second author also discussed a continued fraction identity of Ramanujan and the celebrated Rogers-Ramanujan continued fraction expansion and companions. Some related kinds of the continued fraction identities of Ramanujan were widely studied in the literature, see [1–4, 7–9]. Inspired by the idea in [17], we find that the method can be used for some Rogers-Ramanujan type functions, and we find nice continued fraction expansions for them. For the Rogers-Ramanujan type identitites, Sills [20] gave an annotated and cross-referenced version of Slater's list [21] of Rogers-Ramanujan type identities. When we refer to Slater's list in this paper, we are referring to Sills' version.

In Section 2, we focus on the continued fraction expansions in the following form:

$$\cfrac{z}{a_0 + \cfrac{z}{a_1 + \cfrac{z}{\ddots}}},$$

and discuss this kind of continued fraction expansions for $zF(z)/G(z)$ and its companion $zG(z)/F(z)$ of the Rogers-Ramanujan type, as in (1.1) and (1.2).

In Section 3, we find that our method can be used to give an elementary proof for a continued fraction identity due to Ramanujan which was proved by Andrews in [2].

Section 4 is devoted to expansions of the form:

$$\cfrac{z^2}{a_0 + b_0 z + \cfrac{z^2}{a_1 + b_1 z + \cfrac{z^2}{\ddots}}},$$

and we study this kind ("J-fraction") of continued fraction identities for $z^2F(z)/G(z)$ and $z^2G(z)/F(z)$ of the Rogers-Ramanujan type. We also give some expansions for the q-tangent and q-cotangent functions. Recently, Shin and Zeng in [19] proved a similar kind of the continued fraction expansions which were used to give a unified proof of Josuat-Vergès recent q-analogues of two identities due to Euler and Roselle.

As usual, we follow the notation and terminology in [6]. For $|q| < 1$, the q-shifted factorial is defined by

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k) \quad \text{and} \quad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \text{ for } n \in \mathbb{C}.$$

$F(z)$ & $G(z)$	Expansions	Theorems
$F(z)=\sum_{n\geq 0}\frac{z^nq^{n(n+1)/2}(-q;q)_{n+d}}{(q;q)_n}$ $G(z)=\sum_{n\geq 0}\frac{z^nq^{n(n-1)/2}(-q;q)_{n+d}}{(q;q)_n}$	$zF(z)/G(z)$	Theorem 2.1
$F(z)=\sum_{n\geq 0}\frac{z^nq^{n^2}}{(q^4;q^4)_n}$ $G(z)=\sum_{n\geq 0}\frac{z^nq^{n^2+2n}}{(q^4;q^4)_n}$	$zF(z)/G(z)$ $zG(z)/F(z)$	Theorem 2.2
$F(z)=\sum_{n\geq 0}\frac{z^nq^{n^2}}{(q^4;q^4)_n}$ $G(z)=\sum_{n\geq 0}\frac{z^nq^{n^2+4n}}{(q^4;q^4)_n}$	$zF(z)/G(z)$ $zG(z)/F(z)$	Theorem 2.3
$F(z)=\sum_{n\geq 0}\frac{z^nq^{n^2}(-q;q^2)_n}{(q^2;q^2)_n}$ $G(z)=\sum_{n\geq 0}\frac{z^nq^{n^2+2n}(-q;q^2)_n}{(q^2;q^2)_n}$	$zF(z)/G(z)$ $zG(z)/F(z)$	Theorem 2.4
$F(z)=\sum_{n\geq 0}\frac{z^nq^{2n^2}}{(q;q)_{2n}}$ $G(z)=\sum_{n\geq 0}\frac{z^nq^{2n^2+2n}}{(q;q)_{2n+1}}$	$zF(z)/G(z)$ $zG(z)/F(z)$	Theorem 2.5
$F(z)=\sum_{n\geq 0}\frac{z^nq^{2n^2}}{(q;q)_{2n}}$ $G(z)=\sum_{n\geq 0}\frac{z^nq^{2n^2+2n}}{(q;q)_{2n}}$	$zF(z)/G(z)$ $zG(z)/F(z)$	Theorem 2.6
$F(z)=\sum_{n\geq 0}\frac{z^nq^{n^2}}{(q;q)_{2n}}$ $G(z)=\sum_{n\geq 0}\frac{z^nq^{n^2+2n}}{(q;q)_{2n+1}}$	$zF(z)/G(z)$ $zG(z)/F(z)$	Theorem 2.7
$F(z)=\sum_{n\geq 0}\frac{z^nq^{n^2}}{(q;q)_{2n}}$ $G(z)=\sum_{n\geq 0}\frac{z^nq^{n^2+2n}}{(q;q)_{2n}}$	$zF(z)/G(z)$ $zG(z)/F(z)$	Theorem 2.8
$F(z)=\sum_{n\geq 0}\frac{z^nq^{n^2+n}}{(q;q)_{2n+1}}$ $G(z)=\sum_{n\geq 0}\frac{z^nq^{n^2+n}}{(q;q)_{2n}}$	$zF(z)/G(z)$ $zG(z)/F(z)$	Theorem 2.9
$F(z)=\sum_{n\geq 0}\frac{z^nq^{n^2+2n}(-q;q^2)_n}{(q^4;q^4)_n}$ $G(z)=\sum_{n\geq 0}\frac{z^nq^{n^2}(-q;q^2)_n}{(q^4;q^4)_n}$	$zF(z)/G(z)$	Theorem 2.10

TABLE 1. Continued fraction expansions of $zF(z)/G(z)$ and $zG(z)/F(z)$

The main results in this paper are summarized in Table 1 and Table 2.

We don't claim that neither the method nor all the results are original. However, we tried to be systematic within the context of the Rogers-Ramanujan type identities and we are confident that a fair share of our results are indeed new.

$F(z)$ & $G(z)$	Expansions	Theorems
$F(z)=\sum_{n\geq0}\frac{z^nq^{n^2}}{(q;q)_n}$ $G(z)=\sum_{n\geq0}\frac{z^nq^{n^2+n}}{(q;q)_n}$	$z^2F(z)/G(z)$ $z^2G(z)/F(z)$	Theorem 4.1
$F(z)=\sum_{n\geq0}\frac{z^nq^{n^2}}{(q;q)_n}$ $G(z)=\sum_{n\geq0}\frac{z^nq^{n^2+2n}}{(q;q)_n}$	$z^2G(z)/F(z)$	Theorem 4.2
$F(z)=\sum_{n\geq0}\frac{z^nq^{n^2}}{(q^4;q^4)_n}$ $G(z)=\sum_{n\geq0}\frac{z^nq^{n^2+2n}}{(q^4;q^4)_n}$	$z^2F(z)/G(z)$ $z^2G(z)/F(z)$	Theorem 4.3
$F(z)=\sum_{n\geq0}\frac{z^nq^{n^2}}{(q^4;q^4)_n}$ $G(z)=\sum_{n\geq0}\frac{z^nq^{n^2+4n}}{(q^4;q^4)_n}$	$z^2F(z)/G(z)$ $z^2G(z)/F(z)$	Theorem 4.4
$F(z)=\sum_{n\geq0}\frac{z^nq^{n^2}(-q;q^2)_n}{(q^2;q^2)_n}$ $G(z)=\sum_{n\geq0}\frac{z^nq^{n^2+2n}(-q;q^2)_n}{(q^2;q^2)_n}$	$z^2G(z)/F(z)$	Theorem 4.5
$F(z)=\sum_{n\geq0}\frac{z^nq^{2n^2}}{(q;q)_{2n}}$ $G(z)=\sum_{n\geq0}\frac{z^nq^{2n^2+2n}}{(q;q)_{2n+1}}$	$z^2G(z)/F(z)$ $z^2F(z)/G(z)$	Theorem 4.6 Theorem 4.13
$F(z)=\sum_{n\geq0}\frac{z^nq^{2n^2}}{(q;q)_{2n}}$ $G(z)=\sum_{n\geq0}\frac{z^nq^{2n^2+2n}}{(q;q)_{2n}}$	$z^2G(z)/F(z)$	Theorem 4.7
$F(z)=\sum_{n\geq0}\frac{z^nq^{n^2}}{(q;q)_{2n}}$ $G(z)=\sum_{n\geq0}\frac{z^nq^{n^2+2n}}{(q;q)_{2n+1}}$	$z^2G(z)/F(z)$	Theorem 4.8
$F(z)=\sum_{n\geq0}\frac{z^nq^{n^2}}{(q;q)_{2n}}$ $G(z)=\sum_{n\geq0}\frac{z^nq^{n^2+2n}}{(q;q)_{2n}}$	$z^2F(z)/G(z)$ $z^2G(z)/F(z)$	Theorem 4.9
$F(z)=\sum_{n\geq0}\frac{z^nq^{n^2+n}}{(q;q)_{2n+1}}$ $G(z)=\sum_{n\geq0}\frac{z^nq^{n^2+n}}{(q;q)_{2n}}$	$z^2F(z)/G(z)$	Theorem 4.10
$F(z)=\sum_{n\geq0}\frac{z^n}{(q;q)_{2n+1}}$ $G(z)=\sum_{n\geq0}\frac{z^n}{(q;q)_{2n}}$	$z^2F(z)/G(z)$ $z^2G(z)/F(z)$	Theorem 4.11 Theorem 4.14
$F(z)=\sum_{n\geq0}\frac{z^nq^{n^2}}{(q;q)_{2n+1}}$ $G(z)=\sum_{n\geq0}\frac{z^nq^{n^2}}{(q;q)_{2n}}$	$z^2F(z)/G(z)$	Theorem 4.12
$F(z)=\sum_{n\geq0}\frac{z^nq^{2n^2}}{(q;q)_{2n+1}}$ $G(z)=\sum_{n\geq0}\frac{z^nq^{2n^2}}{(q;q)_{2n}}$	$z^2G(z)/F(z)$	Theorem 4.15
$F(z)=\sum_{n\geq0}\frac{z^nq^{2n}}{(q;q)_{2n+1}}$ $G(z)=\sum_{n\geq0}\frac{z^n}{(q;q)_{2n}}$	$z^2G(z)/F(z)$	Theorem 4.16

TABLE 2. Continued fraction expansions of $z^2F(z)/G(z)$ and $z^2G(z)/F(z)$

2. $zF(z)/G(z)$ AND $zG(z)/F(z)$

For the continued fraction of the form

$$\cfrac{z}{a_0+\cfrac{z}{a_1+\cfrac{z}{\ddots}}},$$

we briefly state the approach as follows.

Let

$$\frac{zF(z)}{G(z)}=\frac{z}{N_0}=\cfrac{z}{a_0+\cfrac{z}{N_1}}=\cfrac{z}{a_0+\cfrac{z}{a_1+\cfrac{z}{N_2}}}=\dots$$

and

$$N_i=\frac{r_i}{s_i}.$$

Then we have

$$N_i=a_i+\frac{z}{N_{i+1}}.$$

That is to say,

$$\frac{r_i}{s_i}=a_i+\frac{zs_{i+1}}{r_{i+1}}.$$

Setting $r_{i+1}=s_i$, we have

$$zs_{i+1}=s_{i-1}-a_is_i, \tag{2.1}$$

where the initial conditions are

$$s_{-1}=G(z)\quad\text{and}\quad s_0=F(z).$$

Therefore, if we guess the number a_k and the power series s_k, and prove the recurrence relation (2.1) by induction, then we prove the continued fraction identities.

Since the proof is a routine computation, we only show the proof for the first example.

2.1. A.8/A.13 type.

First, we consider the continued fraction expansions of the Rogers-Ramanujan type functions in the identities A.8 and A.13 in Slater's list which are stated as follows.

Identity A.8 (Gauss-Lebesgue [13]):

$$\sum_{n=0}^{\infty}\frac{(-q;q)_nq^{n(n+1)/2}}{(q;q)_n}=\frac{(q^4;q^4)_\infty}{(q;q)_\infty}.$$

Identity A.13 (Slater [21]):

$$\sum_{n=0}^{\infty}\frac{(-q;q)_nq^{n(n-1)/2}}{(q;q)_n}=\frac{(q^4;q^4)_\infty}{(q;q)_\infty}+\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}.$$

In the following theorem, we give a more general case by introducing a parameter d.

Theorem 2.1. *Let*

$$F(z) := \sum_{n\geq 0} \frac{z^n q^{n(n+1)/2}(-q;q)_{n+d}}{(q;q)_n}, \qquad G(z) := \sum_{n\geq 0} \frac{z^n q^{n(n-1)/2}(-q;q)_{n+d}}{(q;q)_n}.$$

For the continued fraction expansion

$$\frac{zF(z)}{G(z)} = \cfrac{z}{a_0 + \cfrac{z}{a_1 + \cfrac{z}{\ddots}}}, \tag{2.2}$$

we have

$$a_{2k} = \frac{(-q^{d+1};q)_k}{q^{k(k+2d+3)/2}}, \qquad a_{2k+1} = \frac{q^{k(k+2d+1)/2}}{(-q^{d+1};q)_{k+1}}.$$

$$s_{2k} = q^{k^2+k(d+1)} \sum_{n\geq 0} \frac{z^n q^{n(n+2k+1)/2}(-q^{k+d+1};q)_n(-q;q)_d}{(q;q)_n},$$

$$s_{2k+1} = q^{k(k+1)/2} \sum_{n\geq 0} \frac{z^n q^{n(n+2k+1)/2}(-q;q)_{n+k+d+1}}{(q;q)_n}.$$

Now we give the proof by induction.

Proof. According to the expansion (2.2), we know that $F(z) = s_0$ and $G(z) = s_{-1}$. Therefore, we have

$$\begin{aligned}
s_1 &= \frac{1}{z}(s_{-1} - a_0 s_0) \\
&= \frac{1}{z}\left(\sum_{n\geq 0} \frac{z^n q^{n(n-1)/2}(-q;q)_{n+d}}{(q;q)_n} - \sum_{n\geq 0} \frac{z^n q^{n(n+1)/2}(-q;q)_{n+d}}{(q;q)_n}\right) \\
&= \sum_{n\geq 0} \frac{z^{n-1} q^{n(n-1)/2}(-q;q)_{n+d}}{(q;q)_n}(1-q^n) \\
&= \sum_{n\geq 1} \frac{z^{n-1} q^{n(n-1)/2}(-q;q)_{n+d}}{(q;q)_{n-1}} \\
&= \sum_{n\geq 0} \frac{z^n q^{n(n+1)/2}(-q;q)_{n+d+1}}{(q;q)_n}.
\end{aligned}$$

$$\begin{aligned}
s_2 &= \frac{1}{z}(s_0 - a_1 s_1) \\
&= \frac{1}{z}\left(\sum_{n\geq 0} \frac{z^n q^{n(n+1)/2}(-q;q)_{n+d}}{(q;q)_n} - \frac{1}{(1+q^{d+1})} \sum_{n\geq 0} \frac{z^n q^{n(n+1)/2}(-q;q)_{n+d+1}}{(q;q)_n}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\geq 0} \frac{z^{n-1}q^{n(n+1)/2}(-q;q)_{n+d}}{(q;q)_n(1+q^{d+1})}\left[(1+q^{d+1})-(1+q^{n+d+1})\right]\\
&= \sum_{n\geq 1} \frac{z^{n-1}q^{n(n+1)/2+d+1}(-q;q)_{n+d}}{(q;q)_{n-1}(1+q^{d+1})}\\
&= \sum_{n\geq 0} \frac{z^{n}q^{n(n+3)/2+d+2}(-q;q)_{n+d+1}}{(q;q)_{n}(1+q^{d+1})}.
\end{aligned}
$$

Next, we show for all n that the recurrence relation (2.1) holds. In this case, we need to prove the following two relations:

$$[z^n](s_{2k-1}-a_{2k}s_{2k}) = [z^{n-1}]s_{2k+1}, \tag{2.3}$$

$$[z^n](s_{2k}-a_{2k+1}s_{2k+1}) = [z^{n-1}]s_{2k+2}. \tag{2.4}$$

For the first relation (2.3):

$$
\begin{aligned}
[z^n](s_{2k-1}-a_{2k}s_{2k}) &= q^{k(k-1)/2}\frac{q^{n(n+2k-1)/2}(-q;q)_{n+k+d}}{(q;q)_n}\\
&\quad - \frac{(-q^{d+1};q)_k}{q^{k(k+2d+3)/2}}q^{k^2+k(d+1)}\frac{q^{n(n+2k+1)/2}(-q^{k+d+1};q)_n(-q;q)_d}{(q;q)_n}\\
&= \frac{q^{n(n+1)/2+n(k-1)+k(k-1)/2}(-q;q)_{n+k+d}}{(q;q)_{n-1}}.
\end{aligned}
$$

On the other hand,

$$[z^{n-1}]s_{2k+1} = \frac{q^{n(n+1)/2+n(k-1)+k(k-1)/2}(-q;q)_{n+k+d}}{(q;q)_{n-1}},$$

which is the same.

For the second relation (2.4):

$$
\begin{aligned}
[z^n](s_{2k}-a_{2k+1}s_{2k+1}) &= q^{k^2+k(d+1)}\frac{q^{n(n+2k+1)/2}(-q^{k+d+1};q)_n(-q;q)_d}{(q;q)_n}\\
&\quad - \frac{q^{k(k+2d+1)/2}}{(-q^{d+1};q)_{k+1}}q^{k(k+1)/2}\frac{q^{n(n+2k+1)/2}(-q;q)_{n+k+d+1}}{(q;q)_n}\\
&= \frac{q^{n(n+1)/2+k(n+k+d+2)+d+1}(-q^{k+d+2};q)_{n-1}(-q;q)_d}{(q;q)_{n-1}}.
\end{aligned}
$$

On the other hand,

$$[z^{n-1}]s_{2k+2} = \frac{q^{n(n+1)/2+k(n+k+d+2)+d+1}(-q^{k+d+2};q)_{n-1}(-q;q)_d}{(q;q)_{n-1}},$$

which is the same. $\square$

2.2. **A.16/A.20 type.** The identities A.16 and A.20 in Slater's list are stated as follows.

Identity A.16 (Rogers [18]):

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q^4;q^4)_n}=\frac{1}{(q^2,q^3;q^5)_\infty(-q^2;q^2)_\infty}.$$

Identity A.20 (Rogers [18]):

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^4;q^4)_n}=\frac{1}{(q,q^4;q^5)_\infty(-q^2;q^2)_\infty}.$$

According to the recurrence relation (2.1), we have the following theorem.

Theorem 2.2. *Let*

$$F(z):=\sum_{n\ge 0}\frac{z^nq^{n^2}}{(q^4;q^4)_n},\qquad G(z):=\sum_{n\ge 0}\frac{z^nq^{n^2+2n}}{(q^4;q^4)_n}.$$

(1) For the continued fraction expansion

$$\frac{zF(z)}{G(z)}=\cfrac{z}{a_0+\cfrac{z}{a_1+\cfrac{z}{\ddots}}},$$

we have

$$\begin{aligned}a_{2k}&=(-1)^k(1+q^{4k})q^{2k^2-2k},\ k\ge 1,\quad a_0=1,\\ a_{2k+1}&=(-1)^{k-1}(1+q^{4k+2})q^{-2k^2-4k-1}.\end{aligned}$$

$$\begin{aligned}s_{2k}&=\sum_{n\ge 0}\frac{z^nq^{(n+k)^2}}{(q^2;q^2)_n(-q^2;q^2)_{n+2k}},\\ s_{2k+1}&=(-1)^{k+1}q^{3k^2+4k+1}\sum_{n\ge 0}\frac{z^nq^{n^2+2n(k+1)}}{(q^2;q^2)_n(-q^2;q^2)_{n+2k+1}}.\end{aligned}$$

(2) For the continued fraction expansion

$$\frac{zG(z)}{F(z)}=\cfrac{z}{a_0+\cfrac{z}{a_1+\cfrac{z}{\ddots}}},$$

we have

$$\begin{aligned}a_{2k}&=(-1)^k(1+q^{4k})q^{-2k^2-2k},\ k\ge 1,\quad a_0=1,\\ a_{2k+1}&=(-1)^k(1+q^{4k+2})q^{2k^2-1}.\end{aligned}$$

$$s_{2k} = (-1)^k q^{3k^2+2k} \sum_{n\geq 0} \frac{z^n q^{n^2+2n(k+1)}}{(q^2;q^2)_n(-q^2;q^2)_{n+2k}},$$
$$s_{2k+1} = q^{(k+1)^2} \sum_{n\geq 0} \frac{z^n q^{n^2+2n(k+1)}}{(q^2;q^2)_n(-q^2;q^2)_{n+2k+1}}.$$

We also find a variant of the above theorem.

Theorem 2.3. *Let*

$$F(z) := \sum_{n\geq 0} \frac{z^n q^{n^2}}{(q^4;q^4)_n}, \qquad G(z) := \sum_{n\geq 0} \frac{z^n q^{n^2+4n}}{(q^4;q^4)_n}.$$

(1) For the continued fraction expansion of $zF(z)/G(z)$, we have

$$a_{2k} = (-1)^k(1+q^{4k-2})q^{2k^2-4k+2}, \ k\geq 1, \quad a_0 = 1,$$
$$a_{2k+1} = (-1)^{k-1}(1+q^{4k})q^{-2k^2-2k-1}, \ k\geq 1, \quad a_1 = -\frac{1}{q}.$$

$$s_{2k} = \sum_{n\geq 0} \frac{z^n q^{(n+k)^2}}{(q^2;q^2)_n(-q^2;q^2)_{n+2k-1}}, \ k\geq 1,$$
$$s_{2k+1} = (-1)^{k+1} q^{3k^2+2k+1} \sum_{n\geq 0} \frac{z^n q^{n^2+2n(k+1)}}{(q^2;q^2)_n(-q^2;q^2)_{n+2k}}.$$

(2) For the continued fraction expansion of $zG(z)/F(z)$, we have

$$a_{2k} = (-1)^k(1+q^{4k-2})q^{-2k^2}, \ k\geq 1, \quad a_0 = 1,$$
$$a_{2k+1} = (-1)^k(1+q^{4k})q^{2k^2-2k-1}, \ k\geq 1, \quad a_1 = \frac{1}{q}.$$

$$s_{2k} = (-1)^k q^{3k^2} \sum_{n\geq 0} \frac{z^n q^{n^2+2n(k+1)}}{(q^2;q^2)_n(-q^2;q^2)_{n+2k-1}}, \ k\geq 1,$$
$$s_{2k+1} = \sum_{n\geq 0} \frac{z^n q^{(n+k+1)^2}}{(q^2;q^2)_n(-q^2;q^2)_{n+2k}}.$$

2.3. **A.34/A.36 type.** The identities A.34 and A.36 in Slater's list are stated as follows.

Identity A.34 (Slater [21]): The analytic version of the second Göllnitz-Gordon partition identity.

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} = \frac{1}{(q^3,q^4,q^5;q^8)_\infty}.$$

Identity A.36 (Slater [21]): The analytic version of the first Göllnitz-Gordon partition identity.

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n}=\frac{1}{(q,q^4,q^7;q^8)_\infty}.$$

Theorem 2.4. *Let*

$$F(z):=\sum_{n\geq 0}\frac{z^n q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n},\qquad G(z):=\sum_{n\geq 0}\frac{z^n q^{n^2+2n}(-q;q^2)_n}{(q^2;q^2)_n}.$$

(1) For the continued fraction expansion of $zF(z)/G(z)$, we have

$$a_{2k}=-\frac{(-q;q^2)_k q^{k^2-1}}{m_k m_{k-1}},\ k\geq 1,\quad a_0=1,$$
$$a_{2k+1}=-\frac{m_k^2}{(-q;q^2)_{k+1}q^{k^2+4k+1}}.$$

$$s_{2k}=\sum_{n\geq 0}\frac{z^n q^{(n+k)^2}(-q^{2k+1};q^2)_n(m_k-(1-q^{2n})q^{2k+1}m_{k-1})}{(q^2;q^2)_n},$$
$$s_{2k+1}=-q^{2k^2+4k+1}\sum_{n\geq 0}\frac{z^n q^{n^2+2n(k+1)}(-q;q^2)_{n+k+1}}{(q^2;q^2)_n m_k},$$

where

$$m_k=(-q^3;q^2)_k\sum_{i=0}^{k}\frac{q^{i^2}}{(-q^3;q^2)_i},$$

which satisfy the following recurrence relation:

$$m_k=(1+q^{2k+1})m_{k-1}+q^{k^2}.$$

(2) For the continued fraction expansion of $zG(z)/F(z)$, we have

$$a_{2k}=\frac{(-q;q^2)_k}{q^{k^2+2k}},\qquad a_{2k+1}=\frac{q^{k^2-1}}{(-q;q^2)_{k+1}}.$$

$$s_{2k}=q^{2k^2+2k}\sum_{n\geq 0}\frac{z^n q^{n^2+2n(k+1)}(-q^{2k+1};q^2)_n}{(q^2;q^2)_n},\qquad s_{2k+1}=\sum_{n\geq 0}\frac{z^n q^{(n+k+1)^2}(-q;q^2)_{n+k+1}}{(q^2;q^2)_n}.$$

2.4. **A.38/A.39 type.** The identities A.38 and A.39 in Slater's list are stated as follows.

Identity A.38 (Slater [21]):

$$\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(q;q)_{2n+1}}=\frac{(q^3,q^5,q^8;q^8)_\infty(q^2,q^{14};q^{16})_\infty}{(q;q)_\infty}.$$

Identity A.39 (Jackson [11]):

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q)_{2n}}=\frac{(q,q^7,q^8;q^8)_\infty(q^6,q^{10};q^{16})_\infty}{(q;q)_\infty}.$$

Theorem 2.5. *Let*

$$F(z) := \sum_{n\geq 0} \frac{z^n q^{2n^2}}{(q;q)_{2n}}, \qquad G(z) := \sum_{n\geq 0} \frac{z^n q^{2n^2+2n}}{(q;q)_{2n+1}}.$$

(1) For the continued fraction expansion of $zF(z)/G(z)$, we have

$$a_{2k} = -\frac{(1-q^{4k+1})(1-q^2)^2 q^{2k-2}}{(1-q^{2k-1})(1-q^{2k})(1-q^{2k+1})(1-q^{2k+2})}, \ k\geq 1, \quad a_0 = \frac{1}{1-q},$$
$$a_{2k+1} = -\frac{(1-q^{4k+3})(1-q^{2k+1})^2(1-q^{2k+2})^2}{(1-q^2)^2 q^{6k+2}}.$$

$$s_{2k} = \sum_{n\geq 0} \frac{z^n q^{2(n+k)^2}(1-q^2+q^{2n+2}-q^{2n+2k+1}-q^{2n+2k+2}+q^{2n+4k+3})}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k}(1-q^2)},$$
$$s_{2k+1} = -q^{2k^2+6k+2} \sum_{n\geq 0} \frac{z^n q^{2n^2+4n(k+1)}(1-q^2)}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k+1}(1-q^{2k+1})(1-q^{2k+2})}.$$

(2) For the continued fraction expansion of $zG(z)/F(z)$, we have

$$a_{2k} = \frac{1-q^{4k+1}}{q^{2k}}, \qquad a_{2k+1} = \frac{1-q^{4k+3}}{q^{2k+2}}.$$

$$s_{2k} = q^{2k^2+2k} \sum_{n\geq 0} \frac{z^n q^{2n^2+2n(2k+1)}}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k}}, \qquad s_{2k+1} = \sum_{n\geq 0} \frac{z^n q^{2(n+k+1)^2}}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k+1}}.$$

Theorem 2.6. *Let*

$$F(z) := \sum_{n\geq 0} \frac{z^n q^{2n^2}}{(q;q)_{2n}}, \qquad G(z) := \sum_{n\geq 0} \frac{z^n q^{2n^2+2n}}{(q;q)_{2n}}.$$

(1) For the continued fraction expansion of $zF(z)/G(z)$, we have

$$a_{2k} = -\frac{(1-q^{4k-1})(1-q^2)^2 q^{2k-2}}{(1-q^{2k-1}-q^{2k}+q^{4k-3})(1-q^{2k+1}-q^{2k+2}+q^{4k+1})}, \ k\geq 1, \quad a_0 = 1,$$
$$a_{2k+1} = -\frac{(1-q^{4k+1})(1-q^{2k+1}-q^{2k+2}+q^{4k+1})^2}{(1-q^2)^2 q^{6k+2}}.$$

$$s_{2k} = \sum_{n\geq 0} \frac{z^n q^{2(n+k)^2}(1-q^2+q^{2n+2}-q^{2n+2k+1}-q^{2n+2k+2}+q^{2n+4k+1})}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k-1}(1-q^2)},$$
$$s_{2k+1} = -q^{2k^2+6k+2} \sum_{n\geq 0} \frac{z^n q^{2n^2+4n(k+1)}(1-q^2)}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k}(1-q^{2k+1}-q^{2k+2}+q^{4k+1})}.$$

(2) For the continued fraction expansion of $zG(z)/F(z)$, we have

$$a_{2k} = \frac{1-q^{4k-1}}{q^{2k}}, \ k\geq 1, \quad a_0 = 1, \qquad a_{2k+1} = \frac{1-q^{4k+1}}{q^{2k+2}}.$$

$$s_{2k} = q^{2k^2+2k} \sum_{n\geq 0} \frac{z^n q^{2n^2+2n(2k+1)}}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k-1}}, \qquad s_{2k+1} = \sum_{n\geq 0} \frac{z^n q^{2(n+k+1)^2}}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k}}.$$

2.5. **A.79/A.96 type.** The identities A.79 and A.96 in Slater's list are stated as follows.

Identity A.79 (Rogers [18]):

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_{2n}}=\frac{(q^8,q^{12},q^{20};q^{20})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Identity A.96 (Rogers [18]):

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q;q)_{2n+1}}=\frac{(q^4,q^6,q^{10};q^{10})_\infty(q^2,q^{18};q^{20})_\infty}{(q;q)_\infty}.$$

Theorem 2.7. *Let*

$$F(z):=\sum_{n\geq 0}\frac{z^nq^{n^2}}{(q;q)_{2n}},\qquad G(z):=\sum_{n\geq 0}\frac{z^nq^{n^2+2n}}{(q;q)_{2n+1}}.$$

(1) For the continued fraction expansion of $zF(z)/G(z)$, we have

$$a_{2k}=-\frac{(1-q^{4k+1})q^{2k^2-k-1}}{(1-q^{2k^2-k})(1-q^{2k^2+3k+1})},\qquad a_{2k+1}=-\frac{(1-q^{4k+3})(1-q^{2k^2+3k+1})^2}{q^{2k^2+5k+1}}.$$

$$s_{2k}=\sum_{n\geq 0}\frac{z^nq^{(n+k)^2}(1-q^{2n+2k^2+3k+1})}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k}},$$

$$s_{2k+1}=-q^{3k^2+5k+1}\sum_{n\geq 0}\frac{z^nq^{n^2+2n(k+1)}}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k+1}(1-q^{2k^2+3k+1})}.$$

(2) For the continued fraction expansion of $zG(z)/F(z)$, we have

$$a_{2k}=\frac{1-q^{4k+1}}{q^{2k^2+3k}},\qquad a_{2k+1}=(1-q^{4k+3})q^{2k^2+k-1}.$$

$$s_{2k}=q^{3k^2+3k}\sum_{n\geq 0}\frac{z^nq^{n^2+2n(k+1)}}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k}},\qquad s_{2k+1}=\sum_{n\geq 0}\frac{z^nq^{(n+k+1)^2}}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k+1}}.$$

Theorem 2.8. *Let*

$$F(z):=\sum_{n\geq 0}\frac{z^nq^{n^2}}{(q;q)_{2n}},\quad G(z):=\sum_{n\geq 0}\frac{z^nq^{n^2+2n}}{(q;q)_{2n}}.$$

(1) For the continued fraction expansion of $zF(z)/G(z)$, we have

$$a_{2k}=-(1-q^{4k-1})q^{2k^2-3k+1},\ k\geq 1,\quad a_0=1,\qquad a_{2k+1}=-\frac{1-q^{4k+1}}{q^{2k^2+3k+1}}.$$

$$s_{2k}=\sum_{n\geq 0}\frac{z^nq^{(n+k)^2}}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k-1}},$$

$$s_{2k+1}=-q^{3k^2+3k+1}\sum_{n\geq 0}\frac{z^nq^{n^2+2n(k+1)}}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k}}.$$

(2) For the continued fraction expansion of $zG(z)/F(z)$, we have

$$a_{2k} = \frac{1-q^{4k-1}}{q^{2k^2+k}}, \ k \geq 1, \quad a_0 = 1, \qquad a_{2k+1} = (1-q^{4k+1})q^{2k^2-k-1}.$$

$$s_{2k} = q^{3k^2+k} \sum_{n\geq 0} \frac{z^n q^{n^2+2n(k+1)}}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k-1}},$$

$$s_{2k+1} = \sum_{n\geq 0} \frac{z^n q^{(n+k+1)^2}}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k}}.$$

2.6. **A.94/A.99 type.** The identities A.94 and A.99 in Slater's list are stated as follows.

Identity A.94 (Rogers [18]):

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n+1}} = \frac{(q^3,q^7,q^{10};q^{10})_\infty(q^4,q^{16};q^{20})_\infty}{(q;q)_\infty}.$$

Identity A.99 (Rogers [18]):

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n}} = \frac{(q,q^9,q^{10};q^{10})_\infty(q^8,q^{12};q^{20})_\infty}{(q;q)_\infty}.$$

Theorem 2.9. *Let*

$$F(z) := \sum_{n\geq 0} \frac{z^n q^{n^2+n}}{(q;q)_{2n+1}}, \qquad G(z) := \sum_{n\geq 0} \frac{z^n q^{n^2+n}}{(q;q)_{2n}}.$$

(1) For the continued fraction expansion of $zF(z)/G(z)$, we have

$$a_{2k} = (1-q^{4k+1})q^{2k^2-k}, \qquad a_{2k+1} = \frac{(1-q^{4k+3})}{q^{2k^2+5k+3}}.$$

$$s_{2k} = q^{k^2+k} \sum_{n\geq 0} \frac{z^n q^{n^2+n(2k+1)}}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k}},$$

$$s_{2k+1} = q^{3(k+1)^2} \sum_{n\geq 0} \frac{z^n q^{n^2+n(2k+3)}}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k+1}}.$$

(2) For the continued fraction expansion of $zG(z)/F(z)$, we have

$$a_{2k} = -\frac{(1-q^{4k+1})q^{2k^2-k}}{(1-q^{2k^2-k})(1-q^{2k^2+3k+1})}, \ k \geq 1, \qquad a_0 = \frac{1}{1-q},$$

$$a_{2k+1} = -\frac{(1-q^{4k+3})(1-q^{2k^2+3k+1})^2}{q^{2k^2+5k+3}}.$$

$$s_{2k} = q^{k^2+k} \sum_{n\geq 0} \frac{z^n q^{n^2+n(2k+1)}(1-q^{2n+2k^2+3k+1})}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k}},$$

$$s_{2k+1} = -q^{3(k+1)^2} \sum_{n\geq 0} \frac{z^n q^{n^2+n(2k+3)}}{(q;q)_{2n+1}(q^{2n+3};q^2)_{2k+1}(1-q^{2k^2+3k+1})}.$$

2.7. **A.25 type.** The identity A.25 in Slater's list is stated as follows.

Identity A.25 (Slater [21]):

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^4;q^4)_n} = \frac{(q^3,q^3,q^6;q^6)_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{2.5}$$

Recently, in [14], McLaughlin et al. found a partner to Equation (2.5).

An identity (McLaughlin et al. [14, Eq. (2.7)]):

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^4;q^4)_n} = \frac{(q^6;q^6)_\infty}{(q^4;q^4)_\infty(q^3,q^9;q^{12})_\infty}.$$

Theorem 2.10. *Let*

$$F(z) := \sum_{n\geq 0} \frac{z^n q^{n^2+2n}(-q;q^2)_n}{(q^4;q^4)_n}, \qquad G(z) := \sum_{n\geq 0} \frac{z^n q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n}.$$

For the continued fraction expansion of $zF(z)/G(z)$, we have

$$a_{2k} = \frac{(-q;q^2)_k(1+q^{4k})}{(q;q^2)_k q^{k^2+2k}}, \ k\geq 1, \quad a_0 = 1,$$

$$a_{2k+1} = \frac{(1+q^{4k+2})(q;q^2)_k q^{k^2-1}}{(-q;q^2)_{k+1}}.$$

$$s_{2k} = q^{2k^2+2k} \sum_{n\geq 0} \frac{z^n q^{n^2+2n(k+1)}(-q;q^2)_{n+k}(q;q^2)_k}{(q^2;q^2)_n(-q^2;q^2)_{n+2k}(-q;q^2)_k},$$

$$s_{2k+1} = \sum_{n\geq 0} \frac{z^n q^{(n+k+1)^2}(-q;q^2)_{n+k+1}}{(q^2;q^2)_n(-q^2;q^2)_{n+2k+1}}.$$

3. A continued fraction identity of Ramanujan

In [2], Andrews gave a proof of the following "slightly tricky" continued fraction of Ramanujan

$$\frac{zF(z)}{G(z)} = \cfrac{z}{1+\cfrac{zq}{1+bq+\cfrac{zq^2}{1+bq^2+\cfrac{zq^3}{\ddots}}}}, \tag{3.1}$$

where

$$F(z) = \sum_{n\geq 0} \frac{z^n q^{n^2+n}}{(q;q)_n(-bq;q)_n}, \qquad G(z) = \sum_{n\geq 0} \frac{z^n q^{n^2}}{(q;q)_n(-bq;q)_n}.$$

Later, Bhargava and Adiga proved this identity in [4]. In what follows, applying our method, we can give an elementary proof of this continued fraction identity (3.1).

Proof. In fact, this continued fraction (3.1) fits in the following form:

$$\frac{zF(z)}{G(z)} = \cfrac{z}{a_0 + \cfrac{zq}{a_1 + \cfrac{zq^2}{a_2 + \cfrac{zq^3}{\ddots}}}}. \tag{3.2}$$

We define

$$\frac{zF(z)}{G(z)} = \frac{z}{N_0} = \cfrac{z}{a_0 + \cfrac{zq}{N_1}} = \cfrac{z}{a_0 + \cfrac{zq}{a_1 + \cfrac{zq^2}{N_2}}} = \dots$$

and

$$N_i = \frac{r_i}{s_i}.$$

Then we have

$$N_i = a_i + \frac{zq^{i+1}}{N_{i+1}}.$$

That is to say,

$$\frac{r_i}{s_i} = a_i + \frac{zq^{i+1}s_{i+1}}{r_{i+1}}.$$

Setting $r_{i+1} = s_i$, we have

$$zq^{i+1}s_{i+1} = s_{i-1} - a_i s_i, \tag{3.3}$$

where the initial conditions are

$$s_{-1} = G(z) \quad \text{and} \quad s_0 = F(z).$$

Therefore, if we give the number a_k and the power series s_k, and prove the recurrence relation (3.3) by induction, then we prove the continued fraction identities (3.2).

In this case, we have

$$a_k = 1 + bq^k,\ k \geq 1, \quad a_0 = 1.$$

$$s_k = \sum_{n\geq 0} \frac{z^n q^{n^2+n(k+1)}}{(q;q)_n(-bq;q)_{n+k}}.$$

Then we can prove the continued fraction identity (3.1) by induction. $\square$

4. $z^2F(z)/G(z)$ and $z^2G(z)/F(z)$

In this section, we discuss the following continued fraction expansions:

$$\frac{z^2F(z)}{G(z)} = \cfrac{z^2}{a_0 + b_0z + \cfrac{z^2}{a_1 + b_1z + \cfrac{z^2}{a_2 + b_2z + \cfrac{z^2}{\ddots}}}}, \tag{4.1}$$

$$\frac{z^2G(z)}{F(z)} = \cfrac{z^2}{a_0 + b_0z + \cfrac{z^2}{a_1 + b_1z + \cfrac{z^2}{a_2 + b_2z + \cfrac{z^2}{\ddots}}}}.$$

Here, we take the continued fraction expansion (4.1) as an example to present the method. For the expansion (4.1), let

$$\frac{z^2F(z)}{G(z)} = \frac{z^2}{N_0} = \cfrac{z^2}{a_0 + b_0z + \cfrac{z^2}{N_1}} = \cfrac{z^2}{a_0 + b_0z + \cfrac{z^2}{a_1 + b_1z + \cfrac{z^2}{N_2}}}$$

and

$$N_i = \frac{r_i}{s_i}.$$

Then we have

$$N_i = a_i + b_iz + \frac{z^2}{N_{i+1}}.$$

That is to say,

$$\frac{r_i}{s_i} = a_i + b_iz + \frac{z^2s_{i+1}}{r_{i+1}}.$$

Setting $r_{i+1} = s_i$, we have

$$z^2s_{i+1} = s_{i-1} - (a_i + b_iz)s_i, \tag{4.2}$$

where the initial conditions are

$$s_{-1} = G(z) \quad \text{and} \quad s_0 = F(z).$$

Therefore, if we guess the numbers a_k, b_k, and the power series s_k, and prove the recurrence relation (4.2) by induction, then we prove the continued fraction identity (4.1).

Theorem 4.1. *Let*

$$F(z) := \sum_{n\geq 0} \frac{z^nq^{n^2}}{(q;q)_n}, \qquad G(z) := \sum_{n\geq 0} \frac{z^nq^{n^2+n}}{(q;q)_n}.$$

(1) For the continued fraction expansion of $z^2F(z)/G(z)$, we have

$$a_{2k} = -\frac{(q^2;q^2)_k^4(1-q)^2}{(q;q^2)_k^4(1-q^{2k})(1-q^{2k+1})q^{4k+1}},\ k\geq 1, \quad a_0 = 1,$$
$$a_{2k+1} = \frac{(q;q^2)_k^4(1-q^{2k+1})^3}{(q^2;q^2)_k^4(1-q^{2k+2})(1-q)^2q^{4k+2}}.$$

$$b_{2k} = -\frac{(q^2;q^2)_k^4(1-q)^2(1+q^3-2q^{2k+1}-2q^{2k+3}+q^{4k+2}+q^{4k+3})}{(q;q^2)_k^4(1-q^{2k})^2(1-q^{2k+1})^2q^2},\ k\geq 1, \quad b_0 = -q,$$
$$b_{2k+1} = \frac{(q;q^2)_k^4(1-q^{2k+1})^2(1+q^3-2q^{2k+2}-2q^{2k+4}+q^{4k+4}+q^{4k+5})}{(q^2;q^2)_k^4(1-q^{2k+2})^2(1-q)^2q}.$$

$$s_{2k} = (-1)^k q^{4k^2+3k}\sum_{n\geq 0}\frac{z^nq^{n^2+4nk}(q;q^2)_k(q^3;q^2)_{k-1}(1-q+q^{n+1}-q^{n+2k+1})}{(q;q)_n(q^2;q^2)_k^2},$$
$$s_{2k+1} = (-1)^k q^{4k^2+7k+2}\sum_{n\geq 0}\frac{z^nq^{n^2+2n(2k+1)}(q^2;q^2)_k^2(1-q+q^{n+1}-q^{n+2k+2})}{(q;q)_n(q;q^2)_{k+1}(q^3;q^2)_k}.$$

(2) For the continued fraction expansion of $z^2G(z)/F(z)$, we have

$$a_{2k} = q^{-4k}, \qquad a_{2k+1} = -q^{-4k-3}.$$

$$b_{2k} = 1+q,\ k\geq 1, \quad b_0 = q, \qquad b_{2k+1} = -\frac{1+q}{q}.$$

$$s_{2k} = (-1)^k q^{4k^2+3k}\sum_{n\geq 0}\frac{z^nq^{n^2+n(4k+1)}}{(q;q)_n},$$
$$s_{2k+1} = (-1)^{k-1} q^{4k^2+7k+3}\sum_{n\geq 0}\frac{z^nq^{n^2+n(4k+3)}}{(q;q)_n}.$$

We point out that for the functions $F(z)$ and $G(z)$ in the above theorem, the continued fraction expansions of $zF(z)/G(z)$ and $zG(z)/F(z)$ are provided in [17].

Theorem 4.2. *Let*

$$F(z) := \sum_{n\geq 0}\frac{z^nq^{n^2}}{(q;q)_n}, \qquad G(z) := \sum_{n\geq 0}\frac{z^nq^{n^2+2n}}{(q;q)_n}.$$

For the continued fraction expansion of $z^2G(z)/F(z)$, we have

$$a_{2k} = q^{-4k}, \qquad a_{2k+1} = -q^{-4k-5}.$$

$$b_{2k} = q(1+q), \qquad b_{2k+1} = -\frac{1+q}{q^2}.$$

$$s_{2k} = (-1)^k q^{4k^2+5k}\sum_{n\geq 0}\frac{z^nq^{n^2+2n(2k+1)}}{(q;q)_n},$$

$$s_{2k+1}=(-1)^{k-1}q^{4k^2+9k+5}\sum_{n\ge0}\frac{z^nq^{n^2+4n(k+1)}}{(q;q)_n}.$$

For the functions $F(z)$ and $G(z)$ in the above theorem, the continued fraction expansion of $zG(z)/F(z)$ are provided in [17].

Theorem 4.3. *Let*

$$F(z):=\sum_{n\ge0}\frac{z^nq^{n^2}}{(q^4;q^4)_n},\qquad G(z):=\sum_{n\ge0}\frac{z^nq^{n^2+2n}}{(q^4;q^4)_n}.$$

(1) For the continued fraction expansion of $z^2F(z)/G(z)$, we have

$$a_{2k}=\frac{(-q^6;q^8)_k^2(1+q^{8k})}{(-q^2;q^8)_k^2q^{8k}},\ k\ge1,\quad a_0=1,$$
$$a_{2k+1}=\frac{(-q^2;q^8)_{k+1}^2(1+q^{8k+4})}{(-q^6;q^8)_k^2q^{8k+2}}.$$

$$b_{2k}=\frac{(-q^6;q^8)_k^2(1-q^{16k})}{(-q^2;q^8)_k^2(1+q^{8k+2})(1+q^{8k-2})q^{4k+1}},\ k\ge1,\quad b_0=-\frac{q}{1+q^2},$$
$$b_{2k+1}=\frac{(-q^2;q^8)_{k+1}^2(1-q^{16k+8})}{(-q^6;q^8)_k^2(1+q^{8k+6})(1+q^{8k+2})q^{4k+1}}.$$

$$s_{2k}=q^{8k^2+2k}\sum_{n\ge0}\frac{z^nq^{n^2+4nk}(-q^2;q^8)_k}{(q^2;q^2)_n(-q^2;q^2)_{n+4k}(-q^6;q^8)_k},$$
$$s_{2k+1}=q^{8k^2+10k+2}\sum_{n\ge0}\frac{z^nq^{n^2+2n(2k+1)}(-q^6;q^8)_k}{(q^2;q^2)_n(-q^2;q^2)_{n+4k+2}(-q^2;q^8)_{k+1}}.$$

(2) For the continued fraction expansion of $z^2G(z)/F(z)$, we have

$$a_{2k}=\frac{(-q^6;q^8)_k^2(1+q^{8k})}{(-q^2;q^8)_k^2q^{8k}},\ k\ge1,\quad a_0=1,$$
$$a_{2k+1}=\frac{(-q^2;q^8)_{k+1}^2(1+q^{8k+4})}{(-q^6;q^8)_k^2q^{8k+6}}.$$

$$b_{2k}=\frac{(-q^6;q^8)_k^2(1-q^{16k})}{(-q^2;q^8)_k^2(1+q^{8k+2})(1+q^{8k-2})q^{4k-1}},\ k\ge1,\quad b_0=\frac{q}{1+q^2},$$
$$b_{2k+1}=\frac{(-q^2;q^8)_{k+1}^2(1-q^{16k+8})}{(-q^6;q^8)_k^2(1+q^{8k+6})(1+q^{8k+2})q^{4k+3}}.$$

$$s_{2k}=q^{8k^2+6k}\sum_{n\ge0}\frac{z^nq^{n^2+2n(2k+1)}(-q^2;q^8)_k}{(q^2;q^2)_n(-q^2;q^2)_{n+4k}(-q^6;q^8)_k},$$
$$s_{2k+1}=q^{8k^2+14k+6}\sum_{n\ge0}\frac{z^nq^{n^2+4n(k+1)}(-q^6;q^8)_k}{(q^2;q^2)_n(-q^2;q^2)_{n+4k+2}(-q^2;q^8)_{k+1}}.$$

Theorem 4.4. *Let*

$$F(z) := \sum_{n\geq 0} \frac{z^n q^{n^2}}{(q^4;q^4)_n}, \qquad G(z) := \sum_{n\geq 0} \frac{z^n q^{n^2+4n}}{(q^4;q^4)_n}.$$

(1) For the continued fraction expansion of $z^2F(z)/G(z)$, we have

$$a_{2k} = \frac{(-q^4;q^8)_k^2(1+q^{8k-2})}{(-q^8;q^8)_{k-1}^2 q^{8k-2}}, \ k\geq 1, \quad a_0 = 1,$$
$$a_{2k+1} = \frac{(-q^8;q^8)_k^2(1+q^{8k+2})}{(-q^4;q^8)_k^2 q^{8k+2}}.$$

$$b_{2k} = \frac{(-q^4;q^8)_{k-1}(-q^4;q^8)_k(1-q^{16k-4})}{(-q^8;q^8)_{k-1}(-q^8;q^8)_k q^{4k-1}}, \ k\geq 1, \quad b_0 = -q,$$
$$b_{2k+1} = \frac{(-q^8;q^8)_{k-1}(-q^8;q^8)_k(1-q^{16k+4})}{(-q^4;q^8)_k(-q^4;q^8)_{k+1} q^{4k+1}}, \ k\geq 1, \quad b_1 = \frac{1}{(1+q^4)q}.$$

$$s_{2k} = \frac{q^{8k^2}(-q^8;q^8)_{k-1}}{(-q^4;q^8)_k} \sum_{n\geq 0} \frac{z^n q^{n^2+4nk}}{(q^2;q^2)_n(-q^2;q^2)_{n+4k-1}}, \ k\geq 1,$$
$$s_{2k+1} = \frac{q^{2(2k+1)^2}(-q^4;q^8)_k}{(-q^8;q^8)_k} \sum_{n\geq 0} \frac{z^n q^{n^2+2n(2k+1)}}{(q^2;q^2)_n(-q^2;q^2)_{n+4k+1}}.$$

(2) For the continued fraction expansion of $z^2G(z)/F(z)$, we have

$$a_{2k} = \frac{(-q^4;q^8)_k^2(1+q^{8k-2})}{(-q^8;q^8)_{k-1}^2 q^{8k}}, \ k\geq 1, \quad a_0 = 1,$$
$$a_{2k+1} = \frac{(-q^8;q^8)_k^2(1+q^{8k+2})}{(-q^4;q^8)_k^2 q^{8k+4}}.$$

$$b_{2k} = \frac{(-q^4;q^8)_k^2(1-q^{16k-4})}{(-q^8;q^8)_{k-1}^2(1+q^{8k})(1+q^{8k-4})q^{4k-1}}, \ k\geq 1, \quad b_0 = q,$$
$$b_{2k+1} = \frac{(-q^8;q^8)_{k-1}(-q^8;q^8)_k(1-q^{16k+4})}{(-q^4;q^8)_{k+1}(-q^4;q^8)_k q^{4k+1}}, \ k\geq 1, \quad b_1 = -\frac{q^3}{1+q^4}.$$

$$s_{2k} = q^{8k^2+4k} \sum_{n\geq 0} \frac{z^n q^{n^2+2n(2k+1)}(-q^8;q^8)_{k-1}}{(q^2;q^2)_n(-q^2;q^2)_{n+4k-1}(-q^4;q^8)_k}, \ k\geq 1,$$
$$s_{2k+1} = q^{8k^2+12k+4} \sum_{n\geq 0} \frac{z^n q^{n^2+4n(k+1)}(-q^4;q^8)_k}{(q^2;q^2)_n(-q^2;q^2)_{n+4k+1}(-q^8;q^8)_k}.$$

Theorem 4.5. *Let*

$$F(z) := \sum_{n\geq 0} \frac{z^n q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n}, \qquad G(z) := \sum_{n\geq 0} \frac{z^n q^{n^2+2n}(-q;q^2)_n}{(q^2;q^2)_n}.$$

For the continued fraction expansion of $z^2G(z)/F(z)$, we have

$$a_{2k} = \frac{(-q;q^4)_k}{(-q^3;q^4)_k q^{6k}}, \qquad a_{2k+1} = -\frac{(-q^3;q^4)_k}{(-q;q^4)_{k+1}q^{6k+5}}.$$

$$b_{2k} = \frac{(-q;q^4)_k(1+q^{4k+1}+q^{4k-1})}{(-q^3;q^4)_k q^{2k-1}}, \ k \geq 1, \quad b_0 = q(1+q),$$
$$b_{2k+1} = -\frac{(-q^3;q^4)_k(1+q^{4k+1}+q^{4k+3})}{(-q;q^4)_{k+1}q^{2k+2}}.$$

$$s_{2k} = \frac{(-1)^k q^{6k^2+5k}}{(-q;q^4)_k}\sum_{n\geq 0}\frac{z^n q^{n^2+2n(2k+1)}(-q;q^2)_{n+2k}}{(q^2;q^2)_n},$$
$$s_{2k+1} = \frac{(-1)^{k-1} q^{6k^2+11k+5}}{(-q^3;q^4)_k}\sum_{n\geq 0}\frac{z^n q^{n^2+4n(k+1)}(-q;q^2)_{n+2k+1}}{(q^2;q^2)_n}.$$

Theorem 4.6. *Let*

$$F(z) := \sum_{n\geq 0}\frac{z^n q^{2n^2}}{(q;q)_{2n}}, \qquad G(z) := \sum_{n\geq 0}\frac{z^n q^{2n^2+2n}}{(q;q)_{2n+1}}.$$

For the continued fraction expansion of $z^2G(z)/F(z)$, we have

$$a_{2k} = \frac{(q^7;q^8)_k^2(1-q^{8k+1})}{(q^3;q^8)_k^2 q^{8k}}, \qquad a_{2k+1} = -\frac{(q^3;q^8)_{k+1}^2(1-q^{8k+5})}{(q^7;q^8)_k^2 q^{8k+6}}.$$

$$b_{2k} = \frac{(q^7;q^8)_k^2(1-q^{8k+1})(1+q^2)}{(q^3;q^8)_k^2(1-q^{8k+3})(1-q^{8k-1})}, \ k\geq 1, \quad b_0 = \frac{q^2}{1-q^3},$$
$$b_{2k+1} = -\frac{(q^3;q^8)_{k+1}^2(1-q^{8k+5})(1+q^2)}{(q^7;q^8)_k^2(1-q^{8k+3})(1-q^{8k+7})q^2}.$$

$$s_{2k} = \frac{(-1)^k q^{8k^2+6k}(q^3;q^8)_k}{(q^7;q^8)_k}\sum_{n\geq 0}\frac{z^n q^{2n^2+2n(4k+1)}}{(q;q)_{2n}(q^{2n+1};q^2)_{4k+1}},$$
$$s_{2k+1} = \frac{(-1)^{k-1} q^{8k^2+14k+6}(q^7;q^8)_k}{(q^3;q^8)_{k+1}}\sum_{n\geq 0}\frac{z^n q^{2n^2+2n(4k+3)}}{(q;q)_{2n}(q^{2n+1};q^2)_{4k+3}}.$$

Theorem 4.7. *Let*

$$F(z) := \sum_{n\geq 0}\frac{z^n q^{2n^2}}{(q;q)_{2n}}, \qquad G(z) := \sum_{n\geq 0}\frac{z^n q^{2n^2+2n}}{(q;q)_{2n}}.$$

For the continued fraction expansion of $z^2G(z)/F(z)$, we have

$$a_{2k} = \frac{(q^5;q^8)_k^2(1-q^{8k-1})}{(q;q^8)_k^2 q^{8k}}, \ k\geq 1, \quad a_0 = 1,$$
$$a_{2k+1} = -\frac{(q;q^8)_{k+1}^2(1-q^{8k+3})}{(q^5;q^8)_k^2 q^{8k+6}}.$$

$$b_{2k} = \frac{(q^5;q^8)_k^2(1-q^{8k-1})(1+q^2)}{(q;q^8)_k^2(1-q^{8k+1})(1-q^{8k-3})}, \;\; k \geq 1, \quad b_0 = \frac{q^2}{1-q},$$

$$b_{2k+1} = -\frac{(q;q^8)_{k+1}^2(1-q^{8k+3})(1+q^2)}{(q^5;q^8)_k^2(1-q^{8k+1})(1-q^{8k+5})q^2}.$$

$$s_{2k} = \frac{(-1)^k q^{8k^2+6k}(q;q^8)_k}{(q^5;q^8)_k} \sum_{n\geq 0} \frac{z^n q^{2n^2+2n(4k+1)}}{(q;q)_{2n}(q^{2n+1};q^2)_{4k}},$$

$$s_{2k+1} = \frac{(-1)^{k-1} q^{8k^2+14k+6}(q^5;q^8)_k}{(q;q^8)_{k+1}} \sum_{n\geq 0} \frac{z^n q^{2n^2+2n(4k+3)}}{(q;q)_{2n}(q^{2n+1};q^2)_{4k+2}}.$$

Theorem 4.8. *Let*

$$F(z) := \sum_{n\geq 0} \frac{z^n q^{n^2}}{(q;q)_{2n}}, \qquad G(z) := \sum_{n\geq 0} \frac{z^n q^{n^2+2n}}{(q;q)_{2n+1}}.$$

For the continued fraction expansion of $z^2G(z)/F(z)$, we have

$$a_{2k} = \frac{(q^7;q^8)_k^2(1-q^{8k+1})}{(q^3;q^8)_k^2 q^{8k}}, \qquad a_{2k+1} = -\frac{(q^3;q^8)_{k+1}^2(1-q^{8k+5})}{(q^7;q^8)_k^2 q^{8k+7}}.$$

$$b_{2k} = \frac{(q^7;q^8)_k^2(1-q^{16k+2})}{(q^3;q^8)_k^2(1-q^{8k+3})(1-q^{8k-1})q^{4k-1}}, \;\; k \geq 1, \quad b_0 = \frac{q}{1-q^3},$$

$$b_{2k+1} = -\frac{(q^3;q^8)_k(q^3;q^8)_{k+1}(1-q^{16k+10})}{(q^7;q^8)_k(q^7;q^8)_{k+1}q^{4k+4}}.$$

$$s_{2k} = \frac{(-1)^k q^{8k^2+7k}(q^3;q^8)_k}{(q^7;q^8)_k} \sum_{n\geq 0} \frac{z^n q^{n^2+2n(2k+1)}}{(q^2;q^2)_n(q;q^2)_{n+4k+1}},$$

$$s_{2k+1} = \frac{(-1)^{k-1} q^{8k^2+15k+7}(q^7;q^8)_k}{(q^3;q^8)_{k+1}} \sum_{n\geq 0} \frac{z^n q^{n^2+4n(k+1)}}{(q^2;q^2)_n(q;q^2)_{n+4k+3}}.$$

Theorem 4.9. *Let*

$$F(z) := \sum_{n\geq 0} \frac{z^n q^{n^2}}{(q;q)_{2n}}, \qquad G(z) := \sum_{n\geq 0} \frac{z^n q^{n^2+2n}}{(q;q)_{2n}}.$$

(1) For the continued fraction expansion of $z^2F(z)/G(z)$, we have

$$a_{2k} = -\frac{(q^5;q^8)_k^2(1-q^{8k-1})}{(q;q^8)_k^2 q^{8k-1}}, \;\; k \geq 1, \quad a_0 = 1,$$

$$a_{2k+1} = \frac{(q;q^8)_{k+1}^2(1-q^{8k+3})}{(q^5;q^8)_k^2 q^{8k+2}}.$$

$$b_{2k} = -\frac{(q^5;q^8)_k(q^5;q^8)_{k-1}(1-q^{16k-2})}{(q;q^8)_k(q;q^8)_{k+1}q^{4k}}, k \geq 1, \quad b_0 = -\frac{q}{1-q},$$

$$b_{2k+1} = \frac{(q;q^8)_{k+1}^2(1-q^{16k+6})}{(q^5;q^8)_k(q^5;q^8)_{k+1}(1-q^{8k+1})q^{4k+1}}.$$

$$s_{2k} = \frac{(-1)^k q^{8k^2+k}(q;q^8)_k}{(q^5;q^8)_k} \sum_{n\ge 0} \frac{z^n q^{n^2+4nk}}{(q^2;q^2)_n (q;q^2)_{n+4k}},$$

$$s_{2k+1} = \frac{(-1)^k q^{8k^2+9k+2}(q^5;q^8)_k}{(q;q^8)_{k+1}} \sum_{n\ge 0} \frac{z^n q^{n^2+2n(2k+1)}}{(q^2;q^2)_n (q;q^2)_{n+4k+2}}.$$

(2) For the continued fraction expansion of $z^2G(z)/F(z)$, we have

$$a_{2k} = \frac{(q^5;q^8)_k^2(1-q^{8k-1})}{(q;q^8)_k^2 q^{8k}}, \ k \ge 1, \quad a_0 = 1,$$

$$a_{2k+1} = -\frac{(q;q^8)_{k+1}^2(1-q^{8k+3})}{(q^5;q^8)_k^2 q^{8k+5}}.$$

$$b_{2k} = \frac{(q^5;q^8)_k (q^5;q^8)_{k-1}(1-q^{16k-2})}{(q;q^8)_k (q;q^8)_{k+1} q^{4k-1}}, \ k \ge 1, \quad b_0 = \frac{q}{1-q},$$

$$b_{2k+1} = -\frac{(q;q^8)_k (q;q^8)_{k+1}(1-q^{16k+6})}{(q^5;q^8)_k (q^5;q^8)_{k+1} q^{4k+2}}.$$

$$s_{2k} = \frac{(-1)^k q^{8k^2+5k}(q;q^8)_k}{(q^5;q^8)_k} \sum_{n\ge 0} \frac{z^n q^{n^2+2n(2k+1)}}{(q^2;q^2)_n (q;q^2)_{n+4k}},$$

$$s_{2k+1} = \frac{(-1)^{k-1} q^{8k^2+13k+5}(q^5;q^8)_k}{(q;q^8)_{k+1}} \sum_{n\ge 0} \frac{z^n q^{n^2+4n(k+1)}}{(q^2;q^2)_n (q;q^2)_{n+4k+2}}.$$

Theorem 4.10. *Let*

$$F(z) := \sum_{n\ge 0} \frac{z^n q^{n^2+n}}{(q;q)_{2n+1}}, \qquad G(z) := \sum_{n\ge 0} \frac{z^n q^{n^2+n}}{(q;q)_{2n}}.$$

For the continued fraction expansion of $z^2F(z)/G(z)$, we have

$$a_{2k} = \frac{(q^7;q^8)_k^2(1-q^{8k+1})}{(q^3;q^8)_k^2 q^{8k}}, \qquad a_{2k+1} = -\frac{(q^3;q^8)_{k+1}^2(1-q^{8k+5})}{(q^7;q^8)_k^2 q^{8k+5}}.$$

$$b_{2k} = \frac{(q^7;q^8)_k (q^7;q^8)_{k-1}(1-q^{16k+2})}{(q^3;q^8)_{k+1} (q^3;q^8)_k q^{4k}}, \ k \ge 1, \quad b_0 = \frac{q^3}{1-q^3},$$

$$b_{2k+1} = -\frac{(q^3;q^8)_{k+1} (q^3;q^8)_k(1-q^{16k+10})}{(q^7;q^8)_{k+1} (q^7;q^8)_k q^{4k+3}}.$$

$$s_{2k} = \frac{(-1)^k q^{8k^2+5k}(q^3;q^8)_k}{(q^7;q^8)_k} \sum_{n\ge 0} \frac{z^n q^{n^2+n(4k+1)}}{(q^2;q^2)_n (q;q^2)_{n+4k+1}},$$

$$s_{2k+1} = \frac{(-1)^{k-1} q^{8k^2+13k+5}(q^7;q^8)_k}{(q^3;q^8)_{k+1}} \sum_{n\ge 0} \frac{z^n q^{n^2+n(4k+3)}}{(q^2;q^2)_n (q;q^2)_{n+4k+3}}.$$

For the functions:

$$F(z) := \sum_{n\geq 0} \frac{z^n q^{dn^2}}{(q;q)_{2n+1}}, \tag{4.3}$$

$$G(z) := \sum_{n\geq 0} \frac{z^n q^{dn^2}}{(q;q)_{2n}}, \tag{4.4}$$

the second author in [17] studied the continued fraction expansions of the q-tangent functions $zF(z)/G(z)$ for the cases $d = 0, 1, 2$. Here, we consider the expansions of $z^2F(z)/G(z)$ for the cases $d = 0$ and $d = 1$ in the following two theorems, respectively.

Theorem 4.11. *Let*

$$F(z) := \sum_{n\geq 0} \frac{z^n}{(q;q)_{2n+1}}, \qquad G(z) := \sum_{n\geq 0} \frac{z^n}{(q;q)_{2n}}.$$

For the continued fraction expansion of $z^2F(z)/G(z)$, we have

$$a_{2k} = \frac{(q^7;q^8)_k^2(1-q^{8k+1})}{(q^3;q^8)_k^2 q^{8k}}, \qquad a_{2k+1} = -\frac{(q^3;q^8)_{k+1}^2(1-q^{8k+5})}{(q^7;q^8)_k^2 q^{8k+4}}.$$

$$b_{2k} = \frac{(q^7;q^8)_k^2(1-q^{8k+1})(1+q^2)}{(q^3;q^8)_k^2(1-q^{8k+3})(1-q^{8k-1})q}, \; k \geq 1, \quad b_0 = \frac{q}{1-q^3},$$

$$b_{2k+1} = -\frac{(q^3;q^8)_{k+1}^2(1-q^{8k+5})(1+q^2)}{(q^7;q^8)_k^2(1-q^{8k+7})(1-q^{8k+3})q}.$$

$$s_{2k} = \frac{(-1)^k q^{8k^2+4k}(q^3;q^8)_k}{(q^7;q^8)_k} \sum_{n\geq 0} \frac{z^n}{(q^2;q^2)_n(q;q^2)_{n+4k+1}},$$

$$s_{2k+1} = \frac{(-1)^{k-1} q^{8k^2+12k+4}(q^7;q^8)_k}{(q^3;q^8)_{k+1}} \sum_{n\geq 0} \frac{z^n}{(q^2;q^2)_n(q;q^2)_{n+4k+3}}.$$

Theorem 4.12. *Let*

$$F(z) := \sum_{n\geq 0} \frac{z^n q^{n^2}}{(q;q)_{2n+1}}, \qquad G(z) := \sum_{n\geq 0} \frac{z^n q^{n^2}}{(q;q)_{2n}}.$$

For the continued fraction expansion of $z^2F(z)/G(z)$, we have

$$a_{2k} = \frac{(q^7;q^8)_k^2(1-q^{8k+1})}{(q^3;q^8)_k^2 q^{8k}}, \qquad a_{2k+1} = -\frac{(q^3;q^8)_{k+1}^2(1-q^{8k+5})}{(q^7;q^8)_k^2 q^{8k+3}}.$$

$$b_{2k} = \frac{(q^7;q^8)_k(q^7;q^8)_{k-1}(1-q^{16k+2})}{(q^3;q^8)_{k+1}(q^3;q^8)_k q^{4k+1}}, \; k \geq 1, \quad b_0 = \frac{q^2}{1-q^3},$$

$$b_{2k+1} = -\frac{(q^3;q^8)_{k+1}(q^3;q^8)_k(1-q^{16k+10})}{(q^7;q^8)_{k+1}(q^7;q^8)_k q^{4k+2}}.$$

$$s_{2k} = \frac{(-1)^k q^{8k^2+3k}(q^3;q^8)_k}{(q^7;q^8)_k} \sum_{n\geq 0} \frac{z^n q^{n^2+4nk}}{(q^2;q^2)_n(q;q^2)_{n+4k+1}},$$

$$s_{2k+1}=\frac{(-1)^{k-1}q^{8k^2+11k+3}(q^7;q^8)_k}{(q^3;q^8)_{k+1}}\sum_{n\geq 0}\frac{z^nq^{n^2+2n(2k+1)}}{(q^2;q^2)_n(q;q^2)_{n+4k+3}}.$$

We notice that the above theorem is the same as Theorem 4.10 when replacing $z\to z/q$ there.

For the continued fraction expansion of $z^2F(z)/G(z)$ for the functions (4.3) and (4.4) in the case $d=2$, we cannot find good expressions for b_k.

Finally, we point out that according to the following formula [12]

$$\cfrac{1}{1-\cfrac{c_1z}{1-\cfrac{c_2z}{\ddots}}}=\cfrac{1}{1-c_1z-\cfrac{c_1c_2z^2}{1-(c_2+c_3)z-\cfrac{c_3c_4z^2}{\ddots}}},\tag{4.5}$$

we can make connections between Section 2 and Section 4. By considering the following formula which is a variant of (4.5):

$$\cfrac{1}{a_0+\cfrac{z}{a_1+\cfrac{z}{a_2+\cfrac{z}{\ddots}}}}=\cfrac{1}{A_0+B_0z+\cfrac{z^2}{A_1+B_1z+\cfrac{z^2}{A_2+B_2z+\cfrac{z^2}{\ddots}}}},\tag{4.6}$$

where

$$A_{2k}=a_0\prod_{i=1}^{k}\frac{a_{4i-1}^2a_{4i}}{a_{4i-4}a_{4i-3}^2},\ k\geq 1,\ A_0=a_0,\qquad A_{2k+1}=-\frac{1}{a_0}\prod_{i=0}^{k}\frac{a_{4i+1}^2a_{4i+2}}{a_{4i-2}a_{4i-1}^2},$$

$$B_{2k}=A_{2k}\frac{a_{4k-1}+a_{4k+1}}{a_{4k-1}a_{4k}a_{4k+1}},\ k\geq 1,\ B_0=\frac{1}{a_1},\qquad B_{2k+1}=A_{2k+1}\frac{a_{4k+1}+a_{4k+3}}{a_{4k+1}a_{4k+2}a_{4k+3}},$$

and $a_{-2}=1/a_0$, $a_{-1}=1$, we can get the continued fraction expansions of $z^2F(z)/G(z)$ (resp. $z^2G(z)/F(z)$) in Section 4 from those of $zF(z)/G(z)$ (resp. $zG(z)/F(z)$) in Section 2 as a corollary. However, we found a direct approach more attractive.

In what follows, according to the formula (4.6), we list some continued fraction expansions which do not have good expressions for s_k.

Theorem 4.13. *Let*

$$F(z):=\sum_{n\geq 0}\frac{z^nq^{2n^2}}{(q;q)_{2n}},\qquad G(z):=\sum_{n\geq 0}\frac{z^nq^{2n^2+2n}}{(q;q)_{2n+1}}.$$

For the continued fraction expansion of $z^2F(z)/G(z)$, we have

$$a_{2k}=-\frac{(q^3;q^4)_k^4(q^4;q^4)_k^4(q^7;q^8)_k^2(1-q^{8k+1})(1-q^2)^2}{(q;q^4)_k^4(q^2;q^4)_k^4(q^3;q^8)_k^2(q^{4k-1};q)_4q^{8k+2}},\ k\geq 1,\quad a_0=\frac{1}{1-q},$$

$$a_{2k+1}=\frac{(q;q^4)_{k+1}^4(q^2;q^4)_{k+1}^4(q^3;q^8)_{k+1}^2(1-q^{8k+5})}{(q^3;q^4)_k^4(q^4;q^4)_k^4(q^7;q^8)_k^2(1-q^2)^2(q^{4k+1};q)_4q^{8k+4}}.$$

$$b_{2k} = -\frac{(q^3;q^4)_k^4(q^4;q^4)_k^4(q^7;q^8)_k(q^7;q^8)_{k-1}(1-q^2)^2}{(q;q^4)_k^4(q^2;q^4)_k^4(q^3;q^8)_{k+1}(q^3;q^8)_k(q^{4k-1};q)_4^2q^4}$$
$$\times [q^6(1-q^{8k-1})(1-q^{4k-1})^2(1-q^{4k})^2 + (1-q^{8k+3})(1-q^{4k+1})^2(1-q^{4k+2})^2],$$
$$k \geq 1, \quad b_0 = -\frac{q^2}{(1-q)^2(1-q^3)},$$
$$b_{2k+1} = \frac{(q;q^4)_{k+1}^4(q^2;q^4)_{k+1}^4(q^3;q^8)_{k+1}(q^3;q^8)_k}{(q^3;q^4)_k^4(q^4;q^4)_k^4(q^7;q^8)_{k+1}(q^7;q^8)_k(1-q^2)^2(q^{4k+1};q)_4^2q^2}$$
$$\times [q^6(1-q^{8k+3})(1-q^{4k+1})^2(1-q^{4k+2})^2 + (1-q^{8k+7})(1-q^{4k+3})^2(1-q^{4k+4})^2].$$

The continued fraction expansion of $zF(z)/G(z)$ for the functions $F(z)$ and $G(z)$ in the above theorem is given in Theorem 2.5.

Theorem 4.14. *Let*
$$F(z) := \sum_{n\geq 0} \frac{z^n}{(q;q)_{2n+1}}, \qquad G(z) := \sum_{n\geq 0} \frac{z^n}{(q;q)_{2n}}.$$
For the continued fraction expansion of $z^2G(z)/F(z)$, we have
$$a_{2k} = -\frac{(q^3;q^4)_k^4(q^4;q^4)_k^4(q^7;q^8)_k^2(1-q^{8k+1})(1-q^2)^2}{(q;q^4)_k^4(q^2;q^4)_k^4(q^3;q^8)_k^2(q^{4k-1};q)_4q^{8k+2}}, \ k \geq 1, \quad a_0 = \frac{1}{1-q},$$
$$a_{2k+1} = \frac{(q;q^4)_{k+1}^4(q^2;q^4)_{k+1}^4(q^3;q^8)_{k+1}^2(1-q^{8k+5})}{(q^3;q^4)_k^4(q^4;q^4)_k^4(q^7;q^8)_k^2(1-q^2)^2(q^{4k+1};q)_4q^{8k+2}}.$$
$$b_{2k} = -\frac{(q^3;q^4)_k^4(q^4;q^4)_k^4(q^7;q^8)_k(q^7;q^8)_{k-1}(1-q^2)^2}{(q;q^4)_k^4(q^2;q^4)_k^4(q^3;q^8)_{k+1}(q^3;q^8)_k(q^{4k-1};q)_4^2q^5}$$
$$\times [q^6(1-q^{8k-1})(1-q^{4k-1})^2(1-q^{4k})^2 + (1-q^{8k+3})(1-q^{4k+1})^2(1-q^{4k+2})^2],$$
$$k \geq 1, \quad b_0 = -\frac{q}{(1-q)^2(1-q^3)},$$
$$b_{2k+1} = \frac{(q;q^4)_{k+1}^4(q^2;q^4)_{k+1}^4(q^3;q^8)_{k+1}(q^3;q^8)_k}{(q^3;q^4)_k^4(q^4;q^4)_k^4(q^7;q^8)_{k+1}(q^7;q^8)_k(1-q^2)^2(q^{4k+1};q)_4^2q}$$
$$\times [q^6(1-q^{8k+3})(1-q^{4k+1})^2(1-q^{4k+2})^2 + (1-q^{8k+7})(1-q^{4k+3})^2(1-q^{4k+4})^2].$$

Theorem 4.15. *Let*
$$F(z) := \sum_{n\geq 0} \frac{z^nq^{2n^2}}{(q;q)_{2n+1}}, \qquad G(z) := \sum_{n\geq 0} \frac{z^nq^{2n^2}}{(q;q)_{2n}}.$$
For the continued fraction expansion of $z^2G(z)/F(z)$, we have
$$a_{2k} = -\frac{(q^3;q^4)_k^4(q^4;q^4)_k^4(q^7;q^8)_k^2(1-q^{8k+1})(1-q^2)^2}{(q;q^4)_k^4(q^2;q^4)_k^4(q^3;q^8)_k^2(q^{4k-1};q)_4q^{8k+1}}, \ k \geq 1, \quad a_0 = \frac{1}{1-q},$$
$$a_{2k+1} = \frac{(q;q^4)_{k+1}^4(q^2;q^4)_{k+1}^4(q^3;q^8)_{k+1}^2(1-q^{8k+5})}{(q^3;q^4)_k^4(q^4;q^4)_k^4(q^7;q^8)_k^2(1-q^2)^2(q^{4k+1};q)_4q^{8k+5}}.$$
$$b_{2k} = -\frac{(q^3;q^4)_k^4(q^4;q^4)_k^4(q^7;q^8)_k(q^7;q^8)_{k-1}(1-q^2)^2}{(q;q^4)_k^4(q^2;q^4)_k^4(q^3;q^8)_{k+1}(q^3;q^8)_k(q^{4k-1};q)_4^2q^3}$$

$$\times [q^6(1-q^{8k-1})(1-q^{4k-1})^2(1-q^{4k})^2 + (1-q^{8k+3})(1-q^{4k+1})^2(1-q^{4k+2})^2],$$

$$k \geq 1, \quad b_0 = -\frac{q^3}{(1-q)^2(1-q^3)},$$

$$b_{2k+1} = \frac{(q;q^4)_{k+1}^4 (q^2;q^4)_{k+1}^4 (q^3;q^8)_{k+1} (q^3;q^8)_k}{(q^3;q^4)_k^4 (q^4;q^4)_k^4 (q^7;q^8)_{k+1} (q^7;q^8)_k (1-q^2)^2 (q^{4k+1};q)_4^2 q^3}$$
$$\times [q^6(1-q^{8k+3})(1-q^{4k+1})^2(1-q^{4k+2})^2 + (1-q^{8k+7})(1-q^{4k+3})^2(1-q^{4k+4})^2].$$

Theorem 4.16. *Let*

$$F(z) := \sum_{n\geq 0} \frac{z^n q^{2n}}{(q;q)_{2n+1}}, \qquad G(z) := \sum_{n\geq 0} \frac{z^n}{(q;q)_{2n}}.$$

For the continued fraction expansion of $z^2 G(z)/F(z)$, we have

$$a_{2k} = -\frac{(q^3;q^4)_k^4 (q^4;q^4)_k^4 (q^7;q^8)_k^2 (1-q^{8k+1})(1-q^2)^2}{(q;q^4)_k^4 (q^2;q^4)_k^4 (q^3;q^8)_k^2 (q^{4k-1};q)_4 q^{8k+3}}, \ k \geq 1, \quad a_0 = \frac{1}{1-q},$$

$$a_{2k+1} = \frac{(q;q^4)_{k+1}^4 (q^2;q^4)_{k+1}^4 (q^3;q^8)_{k+1}^2 (1-q^{8k+5})}{(q^3;q^4)_k^4 (q^4;q^4)_k^4 (q^7;q^8)_k^2 (1-q^2)^2 (q^{4k+1};q)_4 q^{8k+1}}.$$

$$b_{2k} = -\frac{(q^3;q^4)_k^4 (q^4;q^4)_k^4 (q^7;q^8)_k (q^7;q^8)_{k-1} (1-q^2)^2}{(q;q^4)_k^4 (q^2;q^4)_k^4 (q^3;q^8)_{k+1} (q^3;q^8)_k (q^{4k-1};q)_4^2 q^6}$$
$$\times [q^6(1-q^{8k-1})(1-q^{4k-1})^2(1-q^{4k})^2 + (1-q^{8k+3})(1-q^{4k+1})^2(1-q^{4k+2})^2],$$

$$k \geq 1, \quad b_0 = -\frac{1}{(1-q)^2(1-q^3)},$$

$$b_{2k+1} = \frac{(q;q^4)_{k+1}^4 (q^2;q^4)_{k+1}^4 (q^3;q^8)_{k+1} (q^3;q^8)_k}{(q^3;q^4)_k^4 (q^4;q^4)_k^4 (q^7;q^8)_{k+1} (q^7;q^8)_k (1-q^2)^2 (q^{4k+1};q)_4^2}$$
$$\times [q^6(1-q^{8k+3})(1-q^{4k+1})^2(1-q^{4k+2})^2 + (1-q^{8k+7})(1-q^{4k+3})^2(1-q^{4k+4})^2].$$

The continued fraction expansions of the q-cotangent functions $zG(z)/F(z)$ for the functions $F(z)$ and $G(z)$ in Theorem 4.14, Theorem 4.15, and Theorem 4.16 are given in [17].

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