Congruences Defined by Languages and Filters

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The usual right congruence $\sim_L$ can be generalized in the following manner:

$$x \sim_L y \iff \{z \mid zx \in L \iff yz \in L\} \in \mathcal{L},$$

where $\mathcal{L}$ is a family of languages.

It turns out to be useful when $\mathcal{L}$ is a filter with an additional property. Furthermore semifilters are introduced and studied. It is also possible to define congruences by filters. Assuming the (right) congruences to have finite index yields a generalization of the regular sets.

1. Introduction and Preliminaries

The well-known mathematical concept filter has been already used in the theory of formal languages (Benda, Bendová, 1976). The same will be done here, but the point of view is another one: Let $L \subseteq \Sigma^*$ and let $\mathcal{L} \subseteq \mathcal{P}(\Sigma^*)$ be a filter with a certain division property (see below). Then by

$$x \sim_{\mathcal{L}} y \iff \{z \mid zx \in L \iff yz \in L\} \in \mathcal{L},$$

a right congruence is defined, which reduces to the well-known right congruence $\sim_L$ of the theory of formal languages by taking $\mathcal{L} = \{\Sigma^*\}$.

A similar concept is used in model theory. (See Bell, Machover (1977, p. 174 ff.).)

With respect to the use of systems $\mathcal{L} \subseteq \mathcal{P}(\Sigma^*)$ in the theory of formal languages compare also (Prodinger, Urbanek, 1979) and (Prodinger, 1979).

In Section 2 necessary and sufficient conditions for a family $\mathcal{L}$ are presented to define a right congruence; appropriate definitions will be given.

In Section 3 the concepts introduced in Section 2 are investigated in detail.

In Section 4 the considerations are extended to the case of congruences.

In Section 5 some generalizations of the family of the regular sets are introduced and closure properties of these families are investigated.

In Section 6 some remarks are made concerning probably the most interesting special case (i.e., if $\mathcal{L}$ is the family of cofinite sets).

Now the essential definitions are given: $\Sigma^*$ denotes the free monoid generated by $\Sigma$ with unit $e$, $\Sigma^* = \Sigma^* - \{e\}$. $\triangle$ denotes the symmetrical difference of two sets; $A \triangle B := (A \triangle B)^c$. $wL = \{z \mid wz \in L\}$ and $L/w = \{z \mid zw \in L\}$. 

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For a formal language $L$ let

$$G_L(x, y) := \{ z \mid zx \in L \iff yz \in L \} = (x \upharpoonright L) \circ (y \upharpoonright L).$$

The right congruence $\sim_L$ is defined by

$$x \sim_L y :\iff G_L(x, y) = \Sigma^*.$$

Finite automata are written as quintuples $(Q, \Sigma, \delta, q_0, F)$. If no final states are considered it will be written $(Q, \Sigma, \delta, q_0)$. The termini state and class are used synonymously. (The state $q$ corresponds to the class $\{w \in \Sigma^* \mid \delta(q_0, w) = q\}.$)

If there is said nothing else, it is assumed that an arbitrary but fixed alphabet $\Sigma$ is given.

It is to be remarked that this paper allows a family $\mathcal{L}$ to be empty.

Concepts of the theory of formal languages not especially described can be found in (Eilenberg, 1974).

2. RIGHT CONGRUENCES AND FILTERS

**Definition 2.1.** A family of languages $\mathcal{L} \subseteq \mathcal{P}(\Sigma^*)$ is called a *filter with division property* (FD), if the following axioms are valid:

- (FD1) $\mathcal{L} \neq \emptyset$
- (FD2) $A, B \in \mathcal{L} \Rightarrow A \cap B \in \mathcal{L}$
- (FD3) $A \in \mathcal{L}, A \subseteq B \Rightarrow B \in \mathcal{L}$
- (FD4) $A \in \mathcal{L}, z \in \Sigma^* \Rightarrow z \backslash A \in \mathcal{L}$.

**Definition 2.2.** A family of languages $\mathcal{L} \subseteq \mathcal{P}(\Sigma^*)$ is called a *semifilter with division property* (SFD), if the following axioms are valid:

- (SFD1) $\Sigma^* \in \mathcal{L}$
- (SFD2) $A, B \in \mathcal{L} \Rightarrow A \circ B \in \mathcal{L}$
- (SFD3) $A \in \mathcal{L}, z \in \Sigma^* \Rightarrow z \backslash A \in \mathcal{L}$.

Each FD is also an SFD: (SFD1) follows from (FD1) and (FD3); (SFD2) follows from (FD2) and (FD3) if $A \circ B = (A \cap B) \cup (A \cup B)^c$ is taken in account.

**Definition 2.3.** $x \sim_{\mathcal{L}, L} y :\iff G_L(x, y) \in \mathcal{L}$.

**Theorem 2.4.** If $\mathcal{L}$ is an SFD then $\sim_{\mathcal{L}, L}$ is a right congruence.
Proof. The reflexivity follows from (SFD1). The symmetry is clear. The transitivity can be seen as follows:

$$G_L(x, z) = (x|L) \circ (z|L) = (x|L) \circ \Sigma^* \circ (z|L)$$

$$= (x|L) \circ (y|L) \circ (z|L) = G_L(x, y) \circ G_L(y, z),$$

and hence (SFD2) can be used.

Now assume $x \sim_{\mathcal{L}, L} y$ and $z \in \Sigma^*$, i.e., $G_L(x, y) = (x|L) \circ (y|L) \in \mathcal{L}$. By

$$(SFD3) \quad \exists ([x|L] \circ (y|L]) = (z'|x|L)) \circ (x'(y|L)) = (xz|L) \circ (yz|L) = G_L(xz, yz) \in \mathcal{L}.$$ 

(The rules for $\circ$, which are used here will be treated in the next section.)

The next theorem can be seen as a conversion of Theorem 2.4:

**Theorem 2.5.** If $|\Sigma| \geq 2$ and $\sim_{\mathcal{L}, L}$ is a right congruence for all $L$, then $\mathcal{L}$ is an SFD.

**Proof.** First the following will be shown: Let $A, B$ be given. Then $x, y, z, L$ can be found, such that

$$(x|L) \circ (y|L) = A \quad \text{and} \quad (y|L) \circ (z|L) = B.$$ 

Let $x = a$, $y = e$, $z = b$. The language $L$ is recursively defined by:

- $\epsilon \notin L$,
- $\sigma \in \Sigma - \{a, b\}$, $w \in \Sigma^* \Rightarrow \sigma w \notin L$,
- $aw \in L \Leftrightarrow [w \in A \leftrightarrow w \in L]$,
- $bw \in L \Leftrightarrow [w \in B \leftrightarrow w \in L]$. 

It is not hard to verify the desired properties.

If (SFD1) does not hold, reflexivity is missing.

If (SFD2) does not hold, i.e., $A, B \in \mathcal{L}$ and $A \circ B \notin \mathcal{L}$, define $x, y, z, L$ as above. Then $x \sim_{\mathcal{L}, L} y, y \sim_{\mathcal{L}, L} z$ but not $x \sim_{\mathcal{L}, L} z$.

If (SFD3) does not hold then $A, z$ exist, such that $A \in \mathcal{L}$, $z|A \notin \mathcal{L}$. Define $x, y, L$ such that $(x|L) \circ (y|L) = A$. Consequently $x \sim_{\mathcal{L}, L} y$ but not $xz \sim_{\mathcal{L}, L} yz$.

It seems to be of a certain interest to take in consideration filters in this context though the filter axioms are stronger than it is necessary; filters are a convenient concept.

Similar to the proof of Theorem 2.4 is the demonstration of

**Theorem 2.6.** If $\mathcal{L}$ is an SFD then by

$$A \sim_{\mathcal{L}} B :\Leftrightarrow A \circ B \in \mathcal{L}$$

an equivalence relation is defined.
Proof. For sake of clarity it will be shown that $A \sim \mathcal{F} B$, $B \sim \mathcal{F} C$ implies $A \sim \mathcal{F} C$.

By the assumptions $A \circ B \in \mathcal{L}$, $B \circ C \in \mathcal{L}$ hold. Hence by (SFD2) $(A \circ B) \circ (B \circ C) = A \circ (B \circ B) \circ C = A \circ \Sigma^* \circ C = A \circ C \in \mathcal{L}$, i.e., $A \sim \mathcal{F} B$.

**Theorem 2.7.** Let be $\mathcal{L}$ an SFD. If $A \sim \mathcal{F} B$ then

$$\sim_{\mathcal{F},A} = \sim_{\mathcal{F},B}.$$

**Proof.** By symmetry it is sufficient to show that $x \sim_{\mathcal{F},A} y$ implies that $x \sim_{\mathcal{F},B} y$.

From $A \circ B \in \mathcal{L}$ follows

$$x \setminus (A \circ B) = (x \setminus A) \circ (x \setminus B) \in \mathcal{L}.$$

By the assumption $(x \setminus A) \circ (y \setminus A) \in \mathcal{L}$. Thus

$$(x \setminus B) \circ (x \setminus A) \circ (y \setminus A) = (x \setminus B) \circ \Sigma^* \circ (y \setminus A) = (x \setminus B) \circ (y \setminus A) \in \mathcal{L}.$$

A similar argumentation gives

$$y \setminus (A \circ B) = (y \setminus A) \circ (y \setminus B) \in \mathcal{L},$$

and therefore

$$(x \setminus B) \circ (y \setminus A) \circ (y \setminus A) \circ (y \setminus B) = (x \setminus B) \circ (y \setminus B) \in \mathcal{L}.$$

(See the next section concerning the rules for $\circ$.)

3. **Properties of FD's and SFD's**

Defining SFD's it is sufficient to substitute (SFD1) by the weaker one

(SFD1') $\mathcal{L} \neq \emptyset$,

since from $A \in \mathcal{L}$ follows $A \circ A = \Sigma^* \in \mathcal{L}$.

It is well-known that $(\Psi(\Sigma^*), \triangle, \cap)$ forms a ring. The valid laws can be reformulated in terms of $\circ$:

- $A \triangle \emptyset = A$, therefore $A \circ \emptyset = A^c$
- $A \triangle A = \emptyset$, therefore $A \circ A = \Sigma^*$
- $A \triangle A^c = \Sigma^*$, therefore $A \circ A^c = \emptyset$. 
\((A \triangle B)^\circ = A \triangle B^\circ\) implies \(A \circ B = A \triangle B^\circ\). Thus \(A \circ \Sigma^* = A \triangle \emptyset = A\).

Hence

\[
A \circ (B \circ C) = A \circ (B \triangle C^\circ) = A^\circ \triangle B \triangle C^\circ \\
= (A \circ B) \triangle C^\circ = (A \circ B) \circ C.
\]

Therefore \((\mathcal{P}(\Sigma^*), \circ)\) is a group, \(\Sigma^*\) being the unit and each element being self-inverse.

\[
(A \circ B) \cup C = [(A \circ B) \cup C]^\circ = [(A \circ B)^\circ \cap C^\circ]^\circ \\
= [(A^\circ \triangle B^\circ) \cap C^\circ]^\circ = [(A^\circ \cap C^\circ) \triangle (B^\circ \cap C^\circ)]^\circ \\
= [(A \cup C)^\circ \triangle (B \cup C)]^\circ = (A \cup C)^\circ \triangle (B \cup C) \\
= (A \cup C) \circ (B \cup C),
\]

thus \((\mathcal{P}(\Sigma^*), \circ, \cup)\) is a ring.

It is evident that \(z \setminus (A \circ B) = (z \setminus A) \circ (z \setminus B)\) holds. For fixed \(z\) the mapping \(A \mapsto z \setminus A\) is an endomorphism of rings.

It is possible to speak of the SFD generated by \(\mathcal{L}\), since \(\mathcal{P}(\Sigma^*)\) is an SFD and arbitrary meets of SFD's are again SFD's.

Now some items to the FD's.

From \(\emptyset \in \mathcal{L}\) follows \(\mathcal{L} = \mathcal{P}(\Sigma^*)\) if \(\mathcal{L}\) is a filter. Therefore especially those FD's are of interest for which \(\emptyset \not\in \mathcal{L}\) does not hold; call them proper.

Again it is possible to speak of the FD generated by \(\mathcal{L}\), and it is interesting, whether or not it is proper.

**Example 1.** Let be \(\mathcal{D} = \{L | L^\circ \text{ is finite}\}\), i.e., \(\mathcal{D}\) is the family of cofinite languages over \(\Sigma\). It is not hard to see that \(\mathcal{D}\) is an FD.

If \(\Sigma = \{a\}\), it is possible to see a subset of \(a^*\) as a 0-1-sequence if one identifies the set with its characteristic function.

As an example, the set \(a(aa)a^*\) corresponds to the 0-1-sequence \(01001001001 \cdots\).

In the sequel \(k\) consecutive 1's in a 0-1-sequence are called 1-block of length \(k\).

**Theorem 3.1.** Let \(\mathcal{L}' \subseteq \mathcal{P}(a^*)\). If \(\mathcal{L}'\) contains an \(A\) with the property that only 1-blocks with a length \(\leq k\) appear, then the FD \(\mathcal{L}\) generated by \(\mathcal{L}'\) is not proper.

**Proof.** Consider \((a \setminus A) \cap A\); this set is in \(\mathcal{L}\) and contains only 1-blocks with a length \(\leq k - 1\). Thus

\[
\emptyset = A \cap (a \setminus A) \cap \cdots \cap (a^k \setminus A) \in \mathcal{L}.
\]

The following theorem is a kind of conversion.
Theorem 3.2. If $A$ contains arbitrary long 1-blocks, then the FD $\mathcal{L}$ generated by $\{A\}$ is proper.

Proof. It is sufficient to show that it is impossible that sets which are in $\mathcal{L}$ by means of (FD2) and (FD4) are the empty set.

Hence it is sufficient to show that always

$$(a^0\setminus A) \cap \cdots \cap (a^n\setminus A) \neq \emptyset.$$ 

Thus it is sufficient to verify that for all $n$

$$(a^0\setminus A) \cap \cdots \cap (a^n\setminus A) \neq \emptyset.$$ 

This is guaranteed by the existence of arbitrary long 1-blocks.

In order to generalize this interpretation as a sequence the following definition is given:

Definition 3.3. \[ \Omega = \Omega(\Sigma) = \{(\omega_n)_{n=0}^{\infty} | \omega_{n+1} = \omega_n \sigma, \sigma \in \Sigma, \omega_0 = \epsilon\} \]

This leads to

Example 2. $\mathcal{L} = \{L | \liminf_{n \to \infty} |\omega_n \cap L|/(n+1) = 1 \text{ for all } \omega \in \Omega\}$ is a proper FD and $\mathcal{D} \subseteq \mathcal{L}$.

If lim inf is replaced by lim sup the generated FD $\mathcal{L}$ is not proper; let $\Sigma = \{a\}$ and construct $A$ as follows: $(\omega = (0, 1, \ldots))$

the $n$-th 1-block is as large as $\frac{|A \cap \omega_n|}{n+1} \geq \frac{1}{n}$,

the $n$-th 0-block is as large as $\frac{|A \cap \omega_n|}{n+1} \leq \frac{1}{n}$,

then $A$ and $A^c$ are in $\mathcal{L}$, and thus $\emptyset = A \cap A^c \in \mathcal{L}$.

It is impossible to dilate Theorem 3.1 for $|\Sigma| \geq 2$:

Theorem 3.4. Let be $\Sigma^* = \{\Sigma^* - Fa^* | F \text{ is a finite set}\}$ and $\mathcal{L} = \{L | \text{ there is an } L' \in \Sigma^* \text{ and } L' \subseteq L\}$, $(a \in \Sigma \text{ fixed})$.

Then $\mathcal{L}$ is a proper FD and there is an $\omega \in \Omega$ such that

$$\limsup_{n \to \infty} \frac{|L \cap \omega_n|}{n+1} = 1 \text{ for all } L \in \mathcal{L}$$

does not hold.

Proof. First it is clear that $\Sigma^* - Fa^*$ can never be $\emptyset$.

It will be shown that for all finite sets $F_1, F_3$ there exists a finite set $F_2$, such that

$$(\Sigma^* - F_1a^*) \cap (\Sigma^* - F_3a^*) \supseteq \Sigma^* - F_2a^*$$
is valid. This is equivalent to

$$F_1a^* \cup F_2a^* \subseteq F_2a^*.$$ 

It is sufficient to choose $$F_2 = F_1 \cup F_2.$$ 

Now let $$F_1$$ be finite and $$z \in \Sigma^*.$$ It will be shown that there exists a finite $$F_2$$ such that

$$z \setminus (\Sigma^* - F_1 a^*) \supseteq \Sigma^* - F_2 a^*$$

holds. This means

$$\Sigma^* - z \setminus (F_1 a^*) \supseteq \Sigma^* - F_2 a^*$$
or

$$z \setminus (F_1 a^*) \subseteq F_2 a^*.$$ 

It is possible to choose $$F_2 = (z \setminus F_1) \cup \{\epsilon\},$$ since from $$w \in z \setminus (F_1 a^*)$$ follows that $$zw \in F_1 a^*.$$ The first case is $$w = w_1 \epsilon$$ and $$zw_1 \in F_1,$$ thus $$w_1 \in z \setminus F_1 ;$$ the second one is $$z = z_1 a^*$$ and $$w = a^*$$, thus $$w \in a^*.$$ 

Let be $$\omega = (\epsilon, a, a^2, \ldots)$$ and $$L = \Sigma^* - a^* \in \mathcal{L}.$$ Then

$$\limsup \frac{|\omega_n \cap L|}{n+1} = \limsup \sup_{n \to \infty} 0 = 0.$$ 

This causes an Example 3.

4. CONGRUENCE RELATIONS AND FILTERS

The syntactic congruence $$\approx_L$$ (cf. Eilenberg (1974)) can be defined as follows:

$$x \approx_L y : \iff \text{for all } u \quad ux \in L \iff uy \in L \quad \text{and} \quad \text{for all } v \quad xv \in L \iff yv \in L.$$ 

This will be generalized in the sequel.

DEFINITION 4.1. A filter (semifilter) $$\mathcal{L}$$ is called FD' (SFD') if it fulfills additionally

$$A \in \mathcal{L}, \ z \in \Sigma^* \Rightarrow A|z \in \mathcal{L}.$$ 

EXAMPLE. $$\mathcal{D}$$ is an FD'.

DEFINITION 4.2. Let $$\mathcal{L}_1$$ be an SFD', $$\mathcal{L}_2$$ an SFD:

$$x \approx_{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_1, \mathcal{L}_2} y : \iff \text{for all } v \quad \text{holds } \{u \mid uxv \in L \iff uyv \in L\} \in \mathcal{L}_1 \quad \text{and} \quad \text{for all } u \quad \text{holds } \{u \mid uxv \in L \iff uyv \in L\} \in \mathcal{L}_2.$$
Theorem 4.3. Under the above mentioned assumptions $\approx_{\mathcal{L}_1, \mathcal{L}_2, L}$ is a congruence relation.

Proof. The proof that $\approx_{\mathcal{L}_1, \mathcal{L}_2, L}$ is an equivalence relation corresponds to the proof of Theorem 2.4.

Assume $x \approx_{\mathcal{L}_1, \mathcal{L}_2, L} y$. It must be shown that for arbitrary $s, t \in \mathcal{L}_1, \mathcal{L}_2, s y$ and $x t \approx_{\mathcal{L}_1, \mathcal{L}_2, L} y t$. By symmetrical argumentations it is sufficient to prove the second part. Let $v$ be arbitrarily chosen. \{\{u | u x v \in L \iff u y v \in L\} \in \mathcal{L}_1\} since this holds for all $v$, especially for $tv$.

Now let $u$ be arbitrarily chosen. \{\{v | u x v \in L \iff u y v \in L\} \in \mathcal{L}_2\}.

5. A Generalization of Regular Sets

It is natural to give the following

Definition 5.1. Let $\mathcal{L}_1$ be an SFD’ and let $\mathcal{L}_2$ be an SFD. Define $\mathcal{R}_{\mathcal{L}_1, \mathcal{L}_2}$ to be the family of all formal languages $L$, such that $\approx_{\mathcal{L}_1, \mathcal{L}_2, L}$ has a finite index.

$\mathcal{R}_{\mathcal{L}_2}$ is the family of all $L$ such that $\sim_{\mathcal{L}_2, L}$ has a finite index.

Obviously the following holds: If $\mathcal{L}_1 \subseteq \mathcal{L}_1', \mathcal{L}_2 \subseteq \mathcal{L}_2'$ then $\mathcal{R}_{\mathcal{L}_1, \mathcal{L}_2} \subseteq \mathcal{R}_{\mathcal{L}_1', \mathcal{L}_2'}$ and $\mathcal{R}_{\mathcal{L}_2} \subseteq \mathcal{R}_{\mathcal{L}_2'}$.

Theorem 5.2. $\mathcal{R}_{\mathcal{L}(\mathcal{L}_1), \mathcal{L}_2} = \mathcal{R}_{\mathcal{L}_2}$.

Proof. The inclusion "$\subseteq"$ is clear.

Now let $(Q, \Sigma, \delta, q_0)$ be the finite automaton without final states corresponding to $\sim_{\mathcal{L}_2, L}$. Furthermore let

$\alpha: \Sigma^* \rightarrow Q^*$ be defined by

$\alpha(v): q \mapsto \delta(q, v)$.

The congruence $\approx$ corresponding to the homomorphism $\alpha$ is a refinement of $\sim_{\mathcal{L}_2, L}$ and has a finite index.

Now assume $w \approx x$, i.e., $\alpha(w) = \alpha(x)$ and let $u$ be an arbitrary element. Then $\alpha(u w) = \alpha(u x)$, i.e., $\delta(q_0, u w) = \delta(q_0, u x)$, thus $u w \sim_{\mathcal{L}_2, L} u x$, hence \{\{v | u x v \in L \iff u y v \in L\} \in \mathcal{L}_2\}; this means $w \approx_{\mathcal{L}(\mathcal{L}_1), \mathcal{L}_2, L} x$. Therefore $\approx_{\mathcal{L}(\mathcal{L}_1), \mathcal{L}_2, L}$ has at most as many classes as $\approx$, i.e., only a finite number of classes.

In the sequel it will be assumed that $\mathcal{L}_1$ is a FD’ and $\mathcal{L}_2$ is a FD.

Lemma 5.3. If $L$ is $\mathcal{L}_1$-regular then $L^c$ is also $\mathcal{L}_1$, $\mathcal{L}_2$-regular.

Proof. Obvious.
Lemma 5.4. If $A, B$ are $\mathcal{L}_1, \mathcal{L}_2$-regular then $A \cap B$ is also $\mathcal{L}_1, \mathcal{L}_2$-regular.

Proof. Define $\approx$ by $\approx_{\mathcal{L}_1, \mathcal{L}_2} \cap \approx_{\mathcal{L}_1, \mathcal{L}_2}$.

Let $x \approx y$ and $u$ an arbitrary element:

$$(ux \setminus A) \circ (uy \setminus A) \in \mathcal{L}_2 \quad \text{and} \quad (ux \setminus B) \circ (uy \setminus B) \in \mathcal{L}_2.$$ 

Therefore

$$[(ux \setminus A) \circ (uy \setminus A)] \cap [(ux \setminus B) \circ (uy \setminus B)] \in \mathcal{L}_2,$$

and this is a subset of

$$(ux \setminus A \cap B) \circ (uy \setminus A \cap B) = [(ux \setminus A) \cap (ux \setminus B)] \circ [(uy \setminus A) \cap (uy \setminus B)],$$

from which follows that the last set is in $\mathcal{L}_2$.

Symmetrically one gets for arbitrary $v$

$$A \cap B \setminus xv \circ (A \cap B \setminus yv) \in \mathcal{L}_1.$$

Thus $x \approx_{\mathcal{L}_1, \mathcal{L}_2} A \cap B \& \approx_{\mathcal{L}_1, \mathcal{L}_2} y$. Therefore $\approx_{\mathcal{L}_1, \mathcal{L}_2} A \cap B$ has not more classes than $\approx$; this yields a finite number of classes.

Corollary 5.5. If $A, B$ are $\mathcal{L}_1, \mathcal{L}_2$-regular then $A \cup B$ is also $\mathcal{L}_1, \mathcal{L}_2$-regular.

As a summary can be stated: ($\emptyset$ is $\mathcal{L}_1, \mathcal{L}_2$-regular).

Theorem 5.6. $\mathcal{L}_1, \mathcal{L}_2$ is a boolean algebra.

6. The Case $\nabla$

The case $\nabla$ seems to be the most interesting one, therefore some remarks concerning this filter will be presented.

If $\approx_{\nabla, L}$ is of finite index, then it is possible to construct the corresponding finite automaton without final states.

It seems suggestive to believe that the following holds: If suitable final states are chosen, a formal language $L'$, "being simpler as $L$ and similar to $L'"$ is obtained. But the following is possible: There are two infinite classes in the minimal automaton of $L$ which coincide with respect to $\approx_{\nabla, L}$. Exactly one of them is a final state; thus in very case $|L' \Delta L| = \infty$. This seems to be not very satisfactory.

It is even possible that this happens considering $\approx_{L}$ which is a refinement of $\approx_{L}$. The two classes coincide with respect to $\approx_{\Psi(\Delta), \nabla, L}$. The language $c^*\{e, a\} \cup c^*\{aa, ba\} c^*$ yields an example.
Furthermore it is false to believe that it is impossible, that the automaton corresponding to $\sim_{\varrho, L}$ contains finite classes; a counter-example is obtained by taking $L = ab^*$. (Two terminal symbols are necessary!)

To obtain the automaton without final states starting with the minimal automaton for $L$ one can proceed as follows:

Assume $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ and let be given the question whether or not the classes $x, y$ coincide with respect to $\sim_{\varrho, L}$. One considers the expression

$$\delta(x, \sigma_1) = \delta(y, \sigma_1) \wedge \cdots \wedge \delta(x, \sigma_n) = \delta(y, \sigma_n)$$

and substitutes distinct $x', y'$ by the analogous expression.

If there appears finally

$$x_1 = x_1 \wedge \cdots \wedge x_s = x_s,$$

the classes coincide, in the other case it happens that after some steps of replacement an expression $\bar{x} = \bar{y}$ will be obtained a second time (a "loop"). Then the classes do not coincide.

A subset $L \subseteq \Sigma^*$ is called \textit{disjunctive} (Shyr, 1977) or \textit{rigid} (Eilenberg, 1976, p. 187) if from $x \sim_{L, y}$ follows $x = y$.

It is natural to give the following

\textbf{DEFINITION 6.1.} $L$ is called $\mathcal{L}_1, \mathcal{L}_2$-disjunctive ($\mathcal{L}$-disjunctive), if $x \sim_{\mathcal{L}_1, \mathcal{L}_2, L, y} (x \sim_{\mathcal{L}, L} y)$ implies $x = y$.

\textbf{THEOREM 6.2.} If a language $L$ is $\{\Sigma^*\}$-disjunctive it is also $\mathcal{B}$-disjunctive.

\textbf{Proof.} Let be $x \neq y$ and $a \in \Sigma$. Because $xa \sim_{L, y} ya$ holds there is a $z \in \Sigma^*$ such that exactly one of the elements $xz, yz$ is in $L$. Thus there is a $z_1 \in \Sigma^+$ such that exactly one of $xz_1, yz_1$ is in $L$. Applying this argumentation to $xz_1, yz_1$ one obtains $z_2 \in \Sigma^+$ etc. Finally one gets an infinite set $\{z_1, z_2, \ldots\}$ such that for all $i$ exactly one of $xz_i, yz_i$ is in $L$. Thus $x \sim_{\varrho, L} y$ is impossible.

The results discussed in this paper seem to be only a small part of problems which can be considered in this context. To give only one example the following open question is cited: Does $\mathcal{R} = \mathcal{B}_{\mathcal{R}}$ hold?

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