SOME DOUBLE BINOMIAL SUMS RELATED TO FIBONACCI, PELL AND GENERALIZED ORDER-k FIBONACCI NUMBERS

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Abstract. We consider some double binomial sums related with the Fibonacci, Pell numbers and a multiple binomial sums related with the generalized order-k Fibonacci numbers. The Lagrange-Bürmann formula and other known techniques are used to prove them.

1. Introduction

The generating function of the Fibonacci numbers $F_n$ is

$$
\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}.
$$

Similarly, the generating function of the Pell numbers $P_n$ is

$$
\sum_{n=0}^{\infty} P_n x^n = \frac{x}{1 - 2x - x^2}.
$$

The generalized order-k Fibonacci numbers $f_n^{(k)}$ are defined by

$$
f_n^{(k)} = \sum_{i=1}^{k} f_{n-i}^{(k)} \quad \text{for} \quad n > k
$$

with initial conditions $f_j^{(k)} = 2^{j-1}$ for $1 \leq j \leq k$.

For example, when $k = 3$, the generalized Fibonacci numbers $f_n^{(3)}$ are reduced to the Tribonacci numbers $T_n$ defined by

$$
T_n = T_{n-1} + T_{n-2} + T_{n-3}
$$

with $T_1 = 1$, $T_2 = 2$ and $T_3 = 4$, for $n > 3$.

For these number sequences, we recall the combinatorial representations due to [2, 3, 5]:

(1.1) $$
\sum_{i=1}^{n} \binom{n-i}{i-1} = F_n,
$$

(1.2) $$
\sum_{i=1}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2i+1} 2^r = P_n,
$$

(1.3) $$
\sum_{i=0}^{n} \sum_{j=0}^{n} \binom{n-i}{j} \binom{n-j}{i} = F_{2n+3}.
$$

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Among the formulas (1.1–1.3), the last formula seems to be different from first two identities just above since it includes double sums, see [2]. The authors of the above cited papers use a combinatorial approach to prove these results. For many similar identities, we refer to [6].

In this paper, we shall derive some new double binomial sums related with the Fibonacci, Pell and generalized order-\(k\) Fibonacci numbers and then use the Lagrange-Bürmann formula and well known other techniques to prove them.

The Lagrange-Bürmann formula is a very useful tool if one knows a series expansion for \(y(x)\) but would like to obtain the series for \(x\) in terms of \(y\). We recall the formula (for details see [1, 4]): Suppose a series for \(y\) in powers of \(x\) is required when \(y = x\Phi(y)\). Assume that \(\Phi\) is analytic in a neighborhood of \(y = 0\) with \(\Phi(0) \neq 0\). Then

\[
x = y/\Phi(y) = \sum_{n=1}^{\infty} a_n y^n, \quad a_1 \neq 0.
\]

Then the two (equivalent) version of the Lagrange(-Bürmann) inversion formula can be written as

\[
F(y) = F(0) + \sum_{n=1}^{\infty} \frac{x^n}{n!} \left[ \frac{d^{n-1}}{dy^{n-1}} \left( F'(y)\Phi^n(y) \right) \right]_{y=0}
\]

or

\[
\frac{F(y)}{1 - x\Phi'(y)} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \left[ \frac{d^n}{dy^n} \left( F(y)\Phi^n(y) \right) \right]_{x=0}.
\]

We would like to rephrase this using the notation of the “coefficient–of” operator:

\[
\frac{F(y)}{1 - x\Phi'(y)} = \sum_{n=0}^{\infty} [y^n] \left( F(y)\Phi^n(y) \right) \cdot x^n;
\]

we will use it in this form.

2. Double Binomial Sums

We start with a result related to Fibonacci numbers:

**Theorem 1.** For \(n > 0\),

\[
F_{4n-1} = \sum_{0 \leq i,j \leq n} \binom{n+i}{2j} \binom{n+j}{2i}.
\]

**Proof.** We start from

\[
[y^{2j}](1+y)^{n+i} = \binom{n+i}{2j}
\]

and compute

\[
S = \sum_{i=0}^{n} (1+y)^{n+i} \binom{n+j}{2i}
\]

\[
= \sum_{i \geq 0} (1+y)^{n+i/2} \binom{n+j}{i} \frac{1+(-1)^i}{2}
\]

\[
= \left[ (1 + \sqrt{1+y})^{j+n} + (1 - \sqrt{1+y})^{j+n} \right] \frac{(1+y)^n}{2}.
\]
here the desired sum takes the form:

\[
\sum_{j=0}^{n} [y^{2j}] \left( (1 + \sqrt{1+y})^{j+n} + (1 - \sqrt{1+y})^{j+n} \right) \frac{(1+y)^n}{2}
\]

\[
= \sum_{j \geq 0} [y^{2j}] \left( (1 + \sqrt{1+y})^{j+n} + (1 - \sqrt{1+y})^{j+n} \right) \frac{(1+y)^n}{2}
\]

\[
= \sum_{j \geq 0} [y^{2j}] \left( (1 + \sqrt{1+y})^{j+n} \frac{(1+y)^n}{2} + \sum_{j \geq 0} [y^{2j}] \left( 1 - \sqrt{1+y} \right)^{j+n} \frac{(1+y)^n}{2} \right)
\]

Let us consider the first sum:

\[
\sum_{j \geq 0} [y^{j}] \left( 1 + \sqrt{1+y} \right)^{j/2+n} (1+y)^n.
\]

This is of the form

\[
\sum_{j \geq 0} [y^{j}] F(y) \Phi(y)^j
\]

with

\[
F(y) = \left( 1 + \sqrt{1+y} \right)^n (1+y)^n \quad \text{and} \quad \Phi(y) = \sqrt{1 + \sqrt{1+y}}.
\]

The Lagrange-Bürmann formula can now be applied to this sum. The general formula is given by

\[
\sum_{j \geq 0} [y^{j}] F(y) \Phi(y)^j \cdot x^j = \frac{F(y)}{1-x\Phi'(y)}.
\]

We need the instance \( x = 1 \) here, and the variables \( x \) and \( y \) are linked via \( y = x\Phi(y) \). Notice that \( \Phi(y) \) must be a power series in \( y \) with a constant term different from zero. Therefore

\[
y = \frac{1 + \sqrt{5}}{2}, \quad F(\alpha) = \left( \frac{7 + 3\sqrt{5}}{2} \right)^n,
\]

\[
\Phi'(\alpha) = \frac{3 - \sqrt{5}}{8}, \quad \frac{1}{1 - \Phi'(\alpha)} = 2 \left( 1 - \frac{1}{\sqrt{5}} \right).
\]

So our evaluation is

\[
2 \left( 1 - \frac{1}{\sqrt{5}} \right) \left( \frac{7 + 3\sqrt{5}}{2} \right)^n.
\]

The second term is

\[
\sum_{j \geq 0} [y^{j}] \left( 1 + \sqrt{1+y} \right)^{j/2+n+1/2} (1+y)^n (-1)^j.
\]

This is the instance \( x = -1 \), which translates to \( y = -1 \) and so the third term is

\[
\frac{F(-1)}{1 + \Phi'(-1)} = 0.
\]
The last sum is
\[ \sum_{j \geq 0} [y^{2j}] (1 - \sqrt{1+y})^{j+n} (1+y)^n = \sum_{j \geq 0} [y^{2j}] y^{j+n} \left( \frac{1 - \sqrt{1+y}}{y} \right)^{j+n} (1+y)^n \]
\[ = \sum_{j \geq 0} [y^j] y^n \left( \frac{1 - \sqrt{1+y}}{y} \right)^{j+n} (1+y)^n. \]

This is again of the form
\[ \sum_{j \geq 0} [y^j] F(y) \Phi(y)^j \]
with
\[ F(y) = \left( \frac{1 - \sqrt{1+y}}{y} \right)^n (1+y)^n \quad \text{and} \quad \Phi(y) = \frac{1 - \sqrt{1+y}}{y}. \]

We need the instance \( x = 1 \) here, and the link is
\[ y = x \left( \frac{1 - \sqrt{1+y}}{y} \right), \]
which means
\[ y = \frac{1 - \sqrt{5}}{2}, \quad F(\beta) = \left( \frac{7 - 3\sqrt{5}}{2} \right)^n \quad \text{and} \quad \frac{1}{1 - \Phi(\alpha)} = 1 + \frac{1}{\sqrt{5}}. \]
So our evaluation is
\[ \left( 1 + \frac{1}{\sqrt{5}} \right) \left( \frac{7 - 3\sqrt{5}}{2} \right)^n. \]

Altogether
\[ \left[ \left( 1 - \frac{1}{\sqrt{5}} \right) \left( \frac{7 + 3\sqrt{5}}{2} \right)^n + \left( 1 + \frac{1}{\sqrt{5}} \right) \left( \frac{7 - 3\sqrt{5}}{2} \right)^n \right] \frac{1}{2} = \frac{\alpha^{4n-1} - \beta^{4n-1}}{\sqrt{5}} = F_{4n-1}, \]
as desired.

**Theorem 2.** For \( n > 0 \),
\[ F_{4n+1} = \sum_{1 \leq i, j \leq n+1} \binom{n+i}{2j-1} \binom{n+j}{2i-1}. \]

**Proof.** Since
\[ [y^{2j-1}] (1+y)^{n+i} = \binom{n+i}{2j-1} \]
and
\[ S = \sum_{i=1}^{n+1} (1+y)^{n+i} \binom{n+j}{2i-1} \]
\[ = \sum_{i \geq 0} (1+y)^{n+i+1/2} \binom{n+j}{i} \frac{1 - (-1)^i}{2} \]
\[ = \left[ (1 + \sqrt{1+y})^{j+n} - (1 - \sqrt{1+y})^{j+n} \right] \frac{(1+y)^{n+1/2}}{2}, \]
here the desired sum takes the form:

\[
\sum_{j=1}^{n+1} [y^{2j-1}] \left( (1 + \sqrt{1+y})^{j+n} - (1 - \sqrt{1+y})^{j+n} \right) \frac{(1+y)^{n+1/2}}{2} \\
= \sum_{j \geq 1} [y^{2j-1}] \left( (1 + \sqrt{1+y})^{j+n} - (1 - \sqrt{1+y})^{j+n} \right) \frac{(1+y)^{n+1/2}}{2} \\
= \sum_{j \geq 1} [y^{2j-1}] \left( 1 + \sqrt{1+y} \right)^{j+n} \frac{(1+y)^{n+1/2}}{2} \\
- \sum_{j \geq 1} [y^{2j-1}] \left( 1 - \sqrt{1+y} \right)^{j+n} \frac{(1+y)^{n+1/2}}{2} \\
= \sum_{j \geq 0} [y^j] \left( 1 + \sqrt{1+y} \right)^{j/2+n+1/2} \frac{(1+y)^{n+1/2}}{2} \left( 1 - (-1)^j \right) \\
- \sum_{j \geq 1} [y^{2j-1}] \left( 1 - \sqrt{1+y} \right)^{j+n} \frac{(1+y)^{n+1/2}}{2}.
\]

Let us start with one term in the above sum:

\[
\sum_{j \geq 0} [y^j] \left( 1 + \sqrt{1+y} \right)^{j/2+n+1/2} (1+y)^{n+1/2}.
\]

This is of the form

\[
\sum_{j \geq 0} [y^j] F(y) \Phi(y)^j
\]

with

\[
F(y) = \left( 1 + \sqrt{1+y} \right)^{n+1/2} (1+y)^{n+1/2} \quad \text{and} \quad \Phi(y) = \sqrt{1 + \sqrt{1+y}}.
\]

This is the instance \( x = 1 \), which translates to

\[
y = \frac{1 + \sqrt{5}}{2}, \quad F(\alpha) = \alpha^{4n+2}
\]

and

\[
\Phi'(\alpha) = \frac{3 - \sqrt{5}}{8}, \quad \frac{1}{1 - \Phi'(\alpha)} = 2 \left( 1 - \frac{1}{\sqrt{5}} \right).
\]

So our evaluation is:

\[
2 \left( 1 - \frac{1}{\sqrt{5}} \right) \alpha^{4n+2}.
\]

The second term is

\[
\sum_{j \geq 0} [y^j] \left( 1 + \sqrt{1+y} \right)^{j/2+n+1/2} (1+y)^{n+1/2}(-1)^j.
\]

This is the instance \( x = -1 \), which translates to \( y = -1 \) and so the second term is

\[
\frac{F(-1)}{1 + \Phi'(-1)} = 0.
\]
Finally the last term is of the form:

\[
\sum_{j \geq 1} [y^{2j-1}] \left( 1 - \sqrt{1 + y} \right)^{j+n} (1 + y)^{n+1/2}
\]

\[
= \sum_{j \geq 1} [y^{2j-1}] y^j \left( \frac{1 - \sqrt{1 + y}}{y} \right)^{j+n} (1 + y)^{n+1/2}
\]

\[
= \sum_{j \geq 0} [y^j] y^{n+1} \left( \frac{1 - \sqrt{1 + y}}{y} \right)^{j+n} (1 + y)^{n+1/2}.
\]

This is of the form:

\[
\sum_{j \geq 0} [y^j] F(y) \Phi(y)^j
\]

with

\[
F(y) = \left( 1 - \sqrt{1 + y} \right)^n (1 + y)^{n+\frac{1}{2}} y \quad \text{and} \quad \Phi(y) = \frac{1 - \sqrt{1 + y}}{y}.
\]

This is the instance \( x = 1 \), which translates to \( y = \beta = \frac{1 - \sqrt{5}}{2} \). Thus

\[
F(\beta) = -\beta^{4n+2}, \quad \Phi'(\beta) = -\frac{1 - \sqrt{5}}{4}, \quad \frac{F(\beta)}{1 - \Phi'(\beta)} = -\left( 1 + \frac{1}{\sqrt{5}} \right) \beta^{4n+2}.
\]

So our evaluation is

\[
\left[ \left( 1 - \frac{1}{\sqrt{5}} \right) \alpha^{4n+2} + \left( 1 + \frac{1}{\sqrt{5}} \right) \beta^{4n+2} \right] \frac{1}{2} = F_{4n+1},
\]

as claimed. \(
\square
\)

**Theorem 3.** For \( n > 0 \),

\[
F_{4n} = \sum_{i=0}^{n} \sum_{j=0}^{n} \left( \frac{n+i}{2j-1} \right) \left( \frac{n+j}{2i} \right),
\]

\[
F_{4n-3} = \sum_{i=0}^{n} \sum_{j=0}^{n} \left( \frac{n+i}{2j+1} \right) \left( \frac{n+j}{2i+1} \right).
\]

Again by using the Lagrange-Bürmann formula, Theorem 3 can be similarly proved.

**Theorem 4.** For \( n > 0 \),

\[
\frac{F_{2n+2} + F_{n+1}}{2} = \sum_{0 \leq i+j \leq n} \left( \frac{n-i}{2j} \right) \left( \frac{n-2j}{i} \right).
\]

**Proof.** First, we replace \( i \) by \( n-i \) and get

\[
\sum_{0 \leq 2j \leq i \leq n} \left( \frac{i}{2j} \right) \left( \frac{n-2j}{i-2j} \right).
\]
Now we compute the generating function of it:
\[
\sum_{n \geq 0} \sum_{0 \leq 2j \leq n} \binom{i}{2j} \binom{n-2j}{i-2j} = \sum_{0 \leq 2j \leq i} \binom{i}{2j} \frac{z^i}{(1-z)^{i+1-2j}}
\]
\[
= \sum_{j \geq 0} z^{2j} (1-z)^{2j} = \frac{1-2z}{(1-z)(1-3z+z^2)}
\]
\[
= \frac{1}{2} \frac{1}{1-z-z^2} + \frac{1}{2} \frac{1}{1-3z-z^2},
\]
which is the generating function of the numbers \((F_{2n+2} + F_{n+1})/2\).

The following results are similar:

**Theorem 5.** For \(n > 0\),
\[
F_{2n} = \sum_{i=1}^{n} \sum_{j=1}^{n} \binom{n-i}{n-j} \binom{n-j}{i-1},
\]
\[
F_{2n-1} = \sum_{0 \leq j \leq n} \binom{n-i}{j} \binom{n-i}{j}.
\]

**Theorem 6.** For \(n > 0\),
\[
F_{2n} + 1 = \sum_{i=0}^{n} F_{2i-1} = \sum_{0 \leq j \leq n} \binom{n-i}{j} \binom{j}{2i}.
\]

**Proof.** Multiplying the right hand side of (2.1) by \(z^n\) and summing over \(n\), we get
\[
S = \sum_{n \geq 0} z^n \sum_{0 \leq i \leq j \leq n} \binom{n-i}{j} \binom{j}{2i} = \sum_{0 \leq i \leq j \leq n} \sum_{h \geq 0} z^{h+i+j} \binom{h+j}{j} \binom{j}{2i}
\]
\[
= \sum_{0 \leq i \leq j} \sum_{h \geq 0} z^{h+i+j} \binom{h+j}{j} \binom{j}{2i} = \sum_{0 \leq i \leq j} \sum_{h \geq 0} z^{h+i+j} \frac{1}{1-(1-z)^{h+i+j+1}}
\]
\[
= \sum_{i \geq 0} \frac{z^{3i}}{(1-2z)^{2i+1}} = \frac{1-2z}{(1-z)(1-3z+z^2)} = \frac{z}{1-3z+z^2} + \frac{1}{1-z},
\]
which is the generating function of the numbers \(F_{2n} + 1\).

For the Pell numbers, we give the following result:

**Theorem 7.** For \(n \geq 0\),
\[
P_{n+1} = \sum_{0 \leq i \leq j \leq n} \binom{n-i}{j} \binom{j}{i}.
\]

**Proof.** Multiplying the right hand side of (2.2) by \(z^n\) and summing over \(n\), we get
\[
S = \sum_{n \geq 0} z^n \sum_{0 \leq i \leq j \leq n} \binom{n-i}{j} \binom{j}{i} = \sum_{0 \leq i \leq j} \sum_{h \geq 0} z^{h+i+j} \binom{h+j}{j} \binom{j}{i}
\]
\[
= \sum_{0 \leq i \leq j} \sum_{h \geq 0} z^{h+i+j} \binom{h+j}{j} \binom{j}{i} = \sum_{0 \leq i \leq j} \sum_{h \geq 0} z^{h+i+j} \frac{1}{1-(1-z)^{h+i+j+1}}
\]
\[ \sum_{0 \leq i \leq j} \frac{z^j}{(1-z)^{j+1}} \binom{j}{i} z^i = \sum_{j \geq 0} \frac{z^j}{(1-z)^{j+1}} (1+z)^j \]
\[ = \frac{1}{1-z} - \frac{1}{1-z(1+z)} = \frac{1}{1-2z-z^2}. \]

This is the generating function of the numbers \( P_{n+1} \). \( \blacksquare \)

Now we give a double sum for the Tribonacci numbers:

**Theorem 8.** For \( n \geq 0 \),
\[ T_n = \sum_{0 \leq j \leq i \leq n} \binom{n-i}{i-j} \binom{i-j}{j}. \]

**Proof.** Consider
\[ \sum_{n \geq 0} T_n z^n = \sum_{0 \leq j \leq i \leq n} z^n \binom{n-i}{i-j} \binom{i-j}{j} = \sum_{0 \leq j \leq i} z^i \binom{i-j}{j} \sum_{h \geq 0} t^h \binom{h}{i-j} \]
\[ = \sum_{0 \leq j \leq i} z^i \binom{i-j}{j} t^{i-j} \sum_{h \geq 0} \frac{z^h}{(1-z)^{h+1}} = \sum_{j \geq 0} \sum_{h \geq 0} z^{h+j} \binom{h+j}{j} \frac{z^h}{(1-z)^{h+1}}. \]

Let \( t = \frac{z^2}{1-z} \), and we continue
\[ \sum_{n \geq 0} T_n z^n = \frac{1}{1-z} \sum_{0 \leq j} z^j \sum_{h \geq 0} \frac{t^h}{(1-t)^{h+1}} = \frac{1}{1-z} \frac{1}{1-z} = \frac{1}{1-z} \frac{1}{1-2z-z} = \frac{1}{1-z-2z-z^2} = \frac{1}{1-z-2z-z^2-z^3}. \]

which is the generating function of the Tribonacci numbers, as expected. So the proof is complete. \( \blacksquare \)

By using the same proof method as in Theorem 8, we get a more general result:

**Theorem 9.** For \( n > 0 \),
\[ f_{n}^{(k)} = \sum_{0 \leq i_k \leq \cdots \leq i_1 \leq n} \binom{n-i_1}{i_1-i_2} \binom{i_1-i_2}{i_2-i_3} \cdots \binom{i_{k-1}-i_k}{i_k} \]
where \( f_{n}^{(k)} \) is the \( n \)-th generalized order-\( k \) Fibonacci number.

**References**


SOME DOUBLE BINOMIAL SUMS


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