PARTIAL SUMS OF HORADAM SEQUENCES: SUM-FREE REPRESENTATIONS VIA GENERATING FUNCTIONS

HELMUT PRODINGER

ABSTRACT. Horadam sequences and their partial sums are computed via generating functions. The results are as simple as possible.

1. Introduction

Horadam sequences [2, 1] $W_n = W_n(a, b; p, q)$ are defined via

$$W_n = pW_{n-1} + qW_{n-2}, \quad n \ge 2, \quad W_0 = a,\ W_1 = b.$$

These numbers are of course a generalization of Fibonacci numbers, Lucas numbers and many others. The characteristic equation

$$X^2 - pX - q = 0$$

is essential, and the two roots are

$$\lambda = \frac{p + \sqrt{p^2 + 4q}}{2}, \quad \mu = \frac{p - \sqrt{p^2 + 4q}}{2}.$$

We define

$$F_n = \frac{\lambda^n - \mu^n}{\lambda - \mu} \quad \text{and} \quad L_n = \lambda^n + \mu^n,$$

as these sequences resemble Fibonacci resp. Lucas numbers, and each solution of the recursion may be expressed as a linear combination of these two sequences. We have $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = p$. For instance

$$W_n = \big(b - \frac{ap}{2}\big)F_n + \frac{a}{2}L_n.$$

The paper [1] concentrates on finding expressions for

$$\sum_{n\le k\le n+m} W_k = \sum_{0\le k\le n+m} W_k - \sum_{0\le k\le n-1} W_k.$$

In the rest of this short paper, we will find simple expressions for

$$S_n := \sum_{0\le k\le n} W_k$$

using generating functions. The results do not contain summations, and can be expressed with the sequences F_n and L_n.

Date: December 22, 2021.

2. Generating functions

Standard computations produce

$$W(z)=\sum_{k\ge0}W_kz^k=\frac{a+z(b-pa)}{1-pz-qz^2};$$

furthermore

$$F(z)=\sum_{k\ge0}F_kz^k=\frac{z}{1-pz-qz^2}\quad\text{and}\quad L(z)=\sum_{k\ge0}L_kz^k=\frac{2-pz}{1-pz-qz^2}.$$

By general principles,

$$\begin{aligned}S(z)=\sum_{k\ge0}S_kz^k&=\frac{1}{1-z}\frac{a+z(b-pa)}{1-pz-qz^2}\\&=\frac{a-pa+b}{1-p-q}\frac{1}{1-z}+\frac{-b-qa+qz(pa-a-b)}{1-p-q}\frac{1}{1-pz-qz^2}\\&=\frac{a-pa+b}{1-p-q}\frac{1}{1-z}\\&\quad-\frac{2qa-pqa+2qb+pb}{2(1-p-q)}\frac{z}{1-pz-qz^2}-\frac{qa+b}{2(1-p-q)}\frac{2-pz}{1-pz-qz^2}.\end{aligned}$$

Reading off the coefficient of z^n on both sides leads to

$$S_n=\frac{a-pa+b}{1-p-q}-\frac{2qa-pqa+2qb+pb}{2(1-p-q)}F_n-\frac{qa+b}{2(1-p-q)}L_n.$$

This answers the finite sum problem addressed in [1] completely, since the answer is

$$S_{n+m}-S_{n-1}=-\frac{2qa-pqa+2qb+pb}{2(1-p-q)}(F_{n+m}-F_{n-1})-\frac{qa+b}{2(1-p-q)}(L_{n+m}-L_{n-1}).$$

References

[1] C. Cooper. Finite sums of consecutive terms of a second order linear recurrence relation. *Integers*, 21, #A114, 2021.

[2] A. F. Horadam. Basic properties of a certain generalized sequence of numbers. *Fibonacci Quart.*, 3, 161–176, 1965.

Helmut Prodinger, Mathematics Department, Stellenbosch University, 7602 Stellenbosch, South Africa.

Email address: `hproding@sun.ac.za`