The number of restricted lattice paths revisited

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Abstract. Ilić and Ilić have recently discussed lattice paths starting and ending at the x-axis which are bounded by two horizontal lines. We establish a link of this to an old paper by Panny and Prodinger where this was already treated.

In [1] the number of random walks from $(0,0)$ to $(2n,0)$ with up-steps and down-steps of one unit each was discussed, under the condition that the walk (path) never touches the line $-h$ and $k$. Here, we want to shed additional light on this, by pointing out that this appeared essentially already in our 1985 paper [2]. Since all this is not complicated, we review the essential steps here. We allow the path to touch $-h$ and $k$, but not $-h-1$ and $k+1$. Further, let $\psi_i(z)$ be the generating function, for $-h \leq i \leq k$, of paths in the sense just described that lead to level $i$. Eventually, we are interested in $\psi_0(z)$.

The following system of linear equations is self-explanatory (and discussed at length in [2]):

\[
\begin{bmatrix}
1 & -z & 0 & \ldots \\
-z & 1 & -z & 0 & \ldots \\
0 & -z & 1 & -z & 0 & \ldots \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
-\ldots & 1 & -z & \ldots & 1
\end{bmatrix}
\begin{bmatrix}
\psi_{-h}(z) \\
\psi_{-h+1}(z) \\
\psi_{-h+2}(z) \\
\psi_0(z) \\
\psi_1(z) \\
\psi_2(z) \\
\psi_k(z)
\end{bmatrix}
= \begin{bmatrix}
0 \\
\vdots \\
\vdots \\
1 \\
\vdots \\
\vdots \\
0
\end{bmatrix}
\]

We use Cramer’s rule to solve this:

\[
\psi_0(z) = \frac{a_{h-k-1}}{a_{h+k}},
\]

where $a_i$ is the determinant of the square matrix with $i+1$ rows and columns. Since $a_i$ satisfies a recursion of second order, it is easy to get

\[
a_i = \frac{1}{1-v^2} \frac{1-v^{2i+4}}{(1+v^2)^{i+1}},
\]

where the substitution $z = v/(1+v^2)$ was used for convenience.

We want to make the parameters $h$ and $k$ explicit and define

\[
f_{h,k}(z) = \psi_0(z).
\]

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The example with lines $−2$ and $5$ corresponds to our $f_{1,4}(z)$. We compute (with Maple):

$$f_{1,4}(z) = 1 + 2z^2 + 5z^4 + 14z^6 + 42z^8 + 131z^{10} + 417z^{12} + 1341z^{14} + 4334z^{16} + 14041z^{18} + 45542z^{20} + 147798z^{22} + 479779z^{24} + 1557649z^{26} + 5057369z^{28} + \ldots,$$

in agreement with [1].

Since (by Cauchy’s integral formula or Lagrange inversion)

$$[z^{2n}]f_{h,k}(z) = [v^n](1 + v)^{2n} \frac{(1 - v^{2h+2})(1 - v^{2k+2})}{(1 - v^{2h+2k+4})} = [v^n](1 + v)^{2n} \frac{(1 - v^{k+1})(1 - v^{k+1})}{(1 - v^{k+2})}$$

$$= \sum_{j\geq 0} \left[ \binom{2n}{n - j(h + k + 2)} - \binom{2n}{n - j(h + k + 2) - h - 1} - \binom{2n}{n - j(h + k + 2) - k - 1} + \binom{2n}{n - (j + 1)(h + k + 2)} \right],$$

we have even an explicit formula. For $h = 1$ and $k = 4$, this gives the sequence

$$1, 2, 5, 14, 42, 131, 417, 1341, 4334, 14041, 45542, 147798, 479779, 1557649, 5057369, \ldots,$$

as expected.

References