THE GENERALIZED q-PILBERT MATRIX

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Abstract. A generalized q-Pilbert matrix from [2] is further generalized, introducing one additional parameter. Explicit formulæ are derived for the LU-decomposition and their inverses, as well as the Cholesky decomposition. The approach is to use q-analysis and to leave the justification of the necessary identities to the q-version of Zeilberger's celebrated algorithm. However, the necessary identities have appeared already in [2] in disguised form, so that no new computations are necessary.

1. Introduction

The Filbert matrix $H_n = \left(\check{h}_{ij}\right)_{i,j=1}^n$ is defined by $\check{h}_{ij} = \frac{1}{F_{i+j-1}}$ as an analogue of the Hilbert matrix where F_n is the nth Fibonacci number. It has been defined and studied by Richardson [4].

In [1], Kılıç and Prodinger studied the generalized matrix with entries $\frac{1}{F_{i+j+r}}$, where $r \geq -1$ is an integer parameter. They gave its LU factorization and, using this, computed its determinant and inverse. Also the Cholesky factorization was derived. After this generalization, Prodinger [3] defined a new generalization of the generalized Filbert matrix by introducing 3 additional parameters. Again, explicit formulæ for the LU-decomposition, their inverses, and the Cholesky factorization were derived.

Recently, in [2], Kılıç and Prodinger give a further generalization of the generalized Filbert Matrix $\mathcal{F}$ with entries $\frac{1}{F_{i+j+r}}$, where $r \geq -1$ is an integer parameter. They define the matrix $\mathcal{Q}$ with entries h_{ij} as follows

$$h_{ij} = \frac{1}{F_{i+j+r}F_{i+j+r+1} \dots F_{i+j+r+k-1}},$$

where $r \geq -1$ is an integer parameter and $k \geq 0$ is an integer parameter.

When $k = 1$, we get the generalized Filbert Matrix $\mathcal{F}$, as studied before. They derive explicit formulæ for the LU-decomposition and their inverses. Again, explicit formulæ for the LU-decomposition, their inverses, and the Cholesky factorization were derived.

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In this paper, we introduce a new kind generalization of the Filbert matrix $\mathcal{F}$ and define the matrix $\mathcal{G}$ with enties g_{ij} by

$$g_{ij} = \frac{1}{F_{\lambda(i+j)+r}F_{\lambda(i+j+1)+r}\dots F_{\lambda(i+j+k-1)+r}},$$

where $r > -1$ and $\lambda > 1$ are integer parameters.

Here we note that the case $\lambda = 1$ was given in [2] so that we shall study the case $\lambda > 1$ throughout this paper. However, all the old results are covered as well, if in some cases the resulting formula is interpreted as a limit.

Our approach will be as follows. We will use the Binet form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q},$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$.

Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$ and as usual for $z > 1$, the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_{(z,y)} = \frac{(q^z;q^y)_n}{(q^z;q^y)_k(q^z;q^y)_{n-k}}$$

and for the case $z = y$, we will denote the Gaussian q-binomial coefficients as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{z} = \frac{(q^z;q^z)_n}{(q^z;q^z)_k(q^z;q^z)_{n-k}}.$$

Here we should note that when $z = 1$, $(q^z;q^y)_n$ would be zero in some cases so that $\begin{bmatrix} n \\ k \end{bmatrix}_{(z,y)}$ would be indefinite. In order to prevent such cases, we will consider the Gaussian q-binomial coefficients for $z > 1$. Furthermore, for the matrix $\mathcal{F}$ and its properties with $z = 1$, we can refer [2].

Considering the definitions of the matrix $\mathcal{G}$ and the q-Pochhammer symbol, we rewrite the matrix $\mathcal{G} = [g_{ij}]$ for $\lambda \geq 1$ as

$$g_{ij} = \mathbf{i}^{k(\lambda(i+j)+r-1)+\frac{\lambda k(k-1)}{2}} q^{-\frac{k}{2}(\lambda(i+j)+r-1)-\frac{\lambda k(k-1)}{4}} \frac{\left(q^{\lambda(i+j)+r};q^{\lambda}\right)_k}{(1-q)^k}.$$

We call the matrix $\mathcal{G}_n$ the *generalized q-Pilbert matrix*. (When $\lambda = 1$, we get the generalized Filbert Matrix $\mathcal{Q}$, as studied before.)

We will derive explicit formulæ for the LU-decomposition and their inverses. Similarly to the results of [1, 2], the size of the matrix does not really matter, and it can be thought about an infinite matrix $\mathcal{G}$ and restrict it whenever necessary to the first n rows resp. columns and write $\mathcal{G}_n$. The entries of the inverse matrix $\mathcal{G}_n^{-1}$ are not closed form expressions, as in our previous paper [1, 2], but can only be given as a (simple) sum. We also provide the Cholesky decomposition.

All the identities we will obtain hold for general q, and results about Fibonacci numbers come out as corollaries for the special choice of q.

Furthermore, we will use *generalized Fibonomial coefficients*

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{(a,b)} = \frac{F_{b(n-1)+a} F_{b(n-2)+a} \dots F_{b(n-k)+a}}{F_a F_{b+a} F_{2b+a} \dots F_{b(k-1)+a}}$$

with $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\}_{(a,b)} = 1$ where F_n is the nth Fibonacci number.

For $a = b$, we denote the generalized Fibonomial coefficents as $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_a$. Especially for $a = b = 1$, the generalized Fibonomial coefficients are reduced to the usual Fibonomial coefficients denoted by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$:

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{F_n F_{n-1} \dots F_{n-k+1}}{F_1 F_2 \dots F_k}.$$

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{(z,y)} = \alpha^{yk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{(z,y)} \quad \text{with} \quad q = -\alpha^{-2}.$$

We will obtain the LU-decomposition $\mathcal{G} = L \cdot U$, where $L = (l_{ij})$ and $U = (u_{ij})$:

Theorem 1. *For* $1 \leq d \leq n$ *we have*

$$l_{n,d} = \mathbf{i}^{\lambda k(d-n)} q^{\lambda \frac{k(n-d)}{2}} \frac{(q^\lambda; q^\lambda)_{n-1} (q^{\lambda(d+1)+r}; q^\lambda)_{d+k-1}}{(q^\lambda; q^\lambda)_{d-1} (q^\lambda; q^\lambda)_{n-d} (q^{\lambda(n+1)+r}; q^\lambda)_{d+k-1}}.$$

As a Fibonacci consequence of Theorem 1, we have

Corollary 1. *For* $1 \leq d \leq n$,

$$l_{n,d} = \left\{ \begin{matrix} n-1 \\ d-1 \end{matrix} \right\}_\lambda \left\{ \begin{matrix} 2d+k \\ d+1 \end{matrix} \right\}_{(r,\lambda)} \left\{ \begin{matrix} n+d+k \\ n+1 \end{matrix} \right\}_{(r,\lambda)}^{-1}.$$

From the Corollary above, we have the following examples: For $\lambda = 2$, $r = -1$,

$$\begin{aligned} l_{n,d} = &\left\{ \begin{matrix} n-1 \\ d-1 \end{matrix} \right\}_2 \left\{ \begin{matrix} n+d+k-2 \\ d+k-1 \end{matrix} \right\}_2 \left\{ \begin{matrix} 4d+2k-3 \\ 2d-1 \end{matrix} \right\} \\ &\times \left\{ \begin{matrix} 2d+k-2 \\ d-1 \end{matrix} \right\}_2^{-1} \left\{ \begin{matrix} 2n+2d+2k-3 \\ 2n-1 \end{matrix} \right\}^{-1}, \end{aligned}$$

and, for $\lambda = 2$, $r = 0$,

$$l_{n,d} = \left\{ \begin{matrix} n-1 \\ d-1 \end{matrix} \right\}_2 \left\{ \begin{matrix} n \\ d \end{matrix} \right\}_2 \left\{ \begin{matrix} n+d+k-1 \\ n-d \end{matrix} \right\}_2^{-1}.$$

Theorem 2. *For $1 \le d \le n$ we have*

$$u_{d,n} = \mathbf{i}^{\lambda\frac{k}{2}(1-k)-\lambda k(n+d)+k-kr} q^{\lambda[\frac{k}{2}\left(d+n-\frac{1}{2}+\frac{k}{2}\right)-d+d^2]+\frac{k(r-1)}{2}-r+dr}(1-q)^k \times \frac{(q^\lambda;q^\lambda)_{d+k-2}(q^\lambda;q^\lambda)_{n-1}}{(q^{\lambda(d+k)+r};q^\lambda)_{d-1}(q^{\lambda(n+1)+r};q^\lambda)_{d+k-1}(q^\lambda;q^\lambda)_{n-d}(q^\lambda;q^\lambda)_{k-1}}.$$

Its Fibonacci Corollary:

Corollary 2. *For $1 \le d \le n$*

$$u_{d,n} = (-1)^{r(d-1)} \begin{Bmatrix} n+d+k \\ n \end{Bmatrix}_{(r;\lambda)}^{-1} \begin{Bmatrix} d+k-2 \\ d-1 \end{Bmatrix}_{\lambda} \begin{Bmatrix} n-1 \\ d-1 \end{Bmatrix}_{\lambda} \times \left(\prod_{t=1}^{d-1} F_{t\lambda}\right)^2 \left(\prod_{t=0}^{2d+k-2} F_{t\lambda+r}\right)^{-1} F_{\lambda n+r}.$$

From the Corollary above, we give the following examples: for $\lambda = 2$, $r = -1$,

$$u_{d,n} = (-1)^{d-1} \begin{Bmatrix} 2n+2d+2k-3 \\ 2n \end{Bmatrix}^{-1} \begin{Bmatrix} n+d+k-2 \\ n-d \end{Bmatrix}_2 \begin{Bmatrix} 2d+k-2 \\ k-1 \end{Bmatrix}_2 \times \left(\prod_{t=1}^{2d-1} F_{2t}\right)\left(\prod_{t=1}^{2d+k-2} F_{2t-1}\right)^{-1} \frac{1}{F_{2n}},$$

and, for $\lambda = 2$, $r = 0$,

$$u_{d,n} = \begin{Bmatrix} 2d+k-2 \\ d-1 \end{Bmatrix}_2^{-1} \begin{Bmatrix} n-1 \\ d-1 \end{Bmatrix}_2 \begin{Bmatrix} n+d+k-1 \\ n+1 \end{Bmatrix}_2^{-1} \left(\prod_{t=1}^{k-1} F_{2t}\right)^{-1} \frac{1}{F_{2n+2}}.$$

We could also determine the inverses of the matrices L and U:

Theorem 3. *For $1 \le d \le n$ we have*

$$l_{n,d}^{-1} = \mathbf{i}^{(\lambda k+2)(d-n)} q^{\frac{\lambda}{2}(d-n)(d-k-n+1)} \frac{(q^\lambda;q^\lambda)_{n-1}(q^{\lambda(d+1)+r};q^\lambda)_{n+k-2}}{(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{n-d}(q^{\lambda(n+1)+r};q^\lambda)_{n+k-2}}.$$

Its Fibonacci Corollary:

Corollary 3. *For $1 \le d \le n$*

$$l_{n,d}^{-1} = \mathbf{i}^{(d-n)(\lambda+d\lambda-n\lambda+2)} \begin{Bmatrix} n-1 \\ d-1 \end{Bmatrix}_{\lambda} \begin{Bmatrix} n+d+k-1 \\ d+1 \end{Bmatrix}_{(r;\lambda)} \begin{Bmatrix} 2n+k-1 \\ n+1 \end{Bmatrix}_{(r;\lambda)}^{-1}.$$

Thus we have the following examples: for $\lambda = 2$, $r = -1$,

$$l_{n,d}^{-1} = (-1)^{d+n} \begin{Bmatrix} 2n+k-3 \\ n-d \end{Bmatrix}_2 \begin{Bmatrix} 2n-1 \\ 2d-1 \end{Bmatrix} \begin{Bmatrix} 4n+2k-5 \\ 2n-2d \end{Bmatrix}^{-1},$$

and, for $\lambda = 2$, $r = 0$,

$$l_{n,d}^{-1} = (-1)^{d+n} \begin{Bmatrix} n-1 \\ d-1 \end{Bmatrix}_2 \begin{Bmatrix} n+d+k-2 \\ d \end{Bmatrix}_2 \begin{Bmatrix} 2n+k-2 \\ n \end{Bmatrix}_2.$$

Theorem 4. *For $1 \le d \le n$ we have*

$$\begin{aligned} u_{d,n}^{-1} &= (-1)^{\frac{\lambda k(d+n)}{2} + \frac{kr}{2} - d + \lambda \frac{k(k-1)}{4} - \frac{k}{2} + n^2} \\ &\quad \times q^{-\lambda \frac{n(n-1)}{2} + r - \lambda \frac{k(d+n)}{2} - \frac{kr}{2} - \lambda nd + \lambda \frac{d(d+1)}{2} - \lambda \frac{k(k-1)}{4} + \frac{k}{2} - rn} \\ &\qquad \times \frac{(q^{\lambda(n+k)+r}; q^\lambda)_n (q^{\lambda(d+1)+r}; q^\lambda)_{n+k-2}}{(q^\lambda; q^\lambda)_{n+k-2} (q^\lambda; q^\lambda)_{d-1} (q^\lambda; q^\lambda)_{n-d}} \frac{(q^\lambda; q^\lambda)_{k-1}}{(1-q)^k}. \end{aligned}$$

And its Fibonacci corollary:

Corollary 4. *For $1 \le d \le n$*

$$\begin{aligned} u_{d,n}^{-1} &= (-1)^{n-d+r(1-n)} \mathbf{i}^{n\lambda(1-n) - d\lambda(2n-1-d)} \left(\prod_{t=0}^{2n+k-2} F_{t\lambda+r} \right) \left(\prod_{t=1}^{2n-2} F_{t\lambda} \right) \\ &\quad \times \begin{Bmatrix} 2n+k \\ n \end{Bmatrix}_{(r,\lambda)} \begin{Bmatrix} n+d+k-1 \\ d+1 \end{Bmatrix}_{(r,\lambda)} \begin{Bmatrix} 2n+k-2 \\ n \end{Bmatrix}_{(r,\lambda)}^{-1} \\ &\qquad \times \begin{Bmatrix} n+k-2 \\ k-1 \end{Bmatrix}_\lambda^{-1} \begin{Bmatrix} n-1 \\ d-1 \end{Bmatrix}_\lambda \begin{Bmatrix} 2n-2 \\ n-1 \end{Bmatrix}_\lambda. \end{aligned}$$

Especially for $\lambda = 2$, $r = -1$,

$$\begin{aligned} u_{d,n}^{-1} &= (-1)^{d+1} \begin{Bmatrix} 2n+2d+2k-5 \\ 2d-2 \end{Bmatrix} \begin{Bmatrix} 2n+k-3 \\ n-d \end{Bmatrix}_2 \begin{Bmatrix} 2n+k-3 \\ k-1 \end{Bmatrix}_2^{-1} \\ &\quad \times \left(\prod_{t=1}^{2n+k-1} F_{2t-1} \right) \left(\prod_{t=1}^{2n-2} F_{2t} \right)^{-1} \frac{1}{F_{2d-1}}, \end{aligned}$$

and, for $\lambda = 2$, $r = 0$,

$$u_{d,n}^{-1} = (-1)^{d+n} \begin{Bmatrix} n+d+k-2 \\ d \end{Bmatrix}_2 \begin{Bmatrix} 2n+k-1 \\ n \end{Bmatrix}_2 \begin{Bmatrix} n \\ d-1 \end{Bmatrix}_2 \left(\prod_{t=1}^{k-1} F_{2t} \right) F_{2d}.$$

As a consequence, we can compute the determinant of $\mathcal{Q}_n$, since it is simply evaluated as $u_{1,1} \cdots u_{n,n}$ (we only state the Fibonacci versions):

Theorem 5.

$$\det \mathcal{G}_n = (-1)^{\frac{r}{2}n(n-1)} \prod_{d=1}^{n} \begin{Bmatrix} 2d+k \\ d \end{Bmatrix}_{(r,\lambda)}^{-1} \begin{Bmatrix} d+k-2 \\ d-1 \end{Bmatrix}_{\lambda} \times \left(\prod_{t=1}^{d-1} F_{t\lambda} \right)^2 \left(\prod_{t=0}^{2d+k-2} F_{t\lambda+r} \right)^{-1} F_{\lambda d+r}.$$

As examples, we have that for $\lambda = 2$ and $r = -1$,

$$\det \mathcal{G}_n = (-1)^{\frac{1}{2}n(n+3)} \prod_{d=1}^{n} \begin{Bmatrix} 4d+2k-3 \\ 2d \end{Bmatrix}^{-1} \begin{Bmatrix} 2d+k-2 \\ k-1 \end{Bmatrix}_{2} \times \left(\prod_{t=1}^{2d-1} F_{2t} \right) \left(\prod_{t=1}^{2d+k-2} F_{2t-1} \right)^{-1} \frac{1}{F_{2d}},$$

and, for $\lambda = 2$, $r = -1$

$$\det \mathcal{G}_n = \left(\prod_{v=1}^{k-1} F_{2v} \right)^{-1} \prod_{d=1}^{n} \begin{Bmatrix} 2d+k-2 \\ d-1 \end{Bmatrix}_{2}^{-1} \begin{Bmatrix} 2d+k-1 \\ d+1 \end{Bmatrix}_{2}^{-1} \frac{1}{F_{2d+2}}.$$

Now we compute the inverse of the matrix $\mathcal{G}$. This time it depends on the dimension, so we compute $(\mathcal{G}_n)^{-1}$.

Theorem 6. *For* $1 \le i, j \le n$*:*

$$\begin{aligned}
&\left((\mathcal{G}_n)^{-1} \right)_{i,k} \\
&= (-1)^{(j-i)-\frac{k}{2}(1-r)-\left(\frac{1-k}{2}-i-j\right)\frac{k\lambda}{2}} q^{r-\left(1-i-j-j^2\right)\frac{\lambda}{2}+\left(\frac{1-k}{2}-i-j\right)\frac{k\lambda}{2}+\frac{k}{2}(1-r)} \\
&\quad \times \frac{\left(q^\lambda; q^\lambda\right)_{k-1}}{(1-q)^k (q^\lambda; q^\lambda)_{j-1} (q^\lambda; q^\lambda)_{i-1} (q^r; q^\lambda)_{i+1} (q^r; q^\lambda)_{j+1}} \\
&\quad \times \sum_{\max\{i,j\} \le h \le n} \frac{\left(q^r; q^\lambda\right)_{h+k+i-1} (q^r; q^\lambda)_{h+1} (q^r; q^\lambda)_{h+k+j-1} (q^\lambda; q^\lambda)_{h-1}}{(q^r; q^\lambda)_{h+k} (q^\lambda; q^\lambda)_{h+k-2} (q^\lambda; q^\lambda)_{h-i} (q^\lambda; q^\lambda)_{h-j}} \\
&\quad \times \left(1 - q^{\lambda(2h+k-1)+r}\right) q^{-hj\lambda-hr-ih\lambda}.
\end{aligned}$$

Finally, we provide the Cholesky decomposition.

Theorem 7. *For $i, j \geq 1$:*

$$\begin{aligned}\mathcal{C}_{i,j} = &\frac{(q^{\lambda};q^{\lambda})_{i-1}(1-q)^{\frac{k}{2}}}{(q^{\lambda(i+1)+r};q^{\lambda})_{j+k-1}(q^{\lambda};q^{\lambda})_{i-j}} \\ &\times \mathbf{i}^{-\lambda\frac{k^2}{4}+\lambda\frac{k}{4}+\frac{k}{2}+\frac{3rk}{2}-\lambda ik} q^{\lambda\frac{j(j-1)}{2}+\lambda\frac{ki}{2}+\lambda\frac{k^2}{8}-\lambda\frac{k}{8}-\frac{k}{4}+\frac{rj}{2}+\frac{kr}{4}-\frac{r}{2}} \\ &\times \sqrt{\frac{(1-q^{\lambda(2j+k-1)+r})(q^{\lambda};q^{\lambda})_{j+k-2}(q^{\lambda(j+1)+r};q^{\lambda})_{k-1}}{(q^{\lambda};q^{\lambda})_{k-1}(q^{\lambda};q^{\lambda})_{j-1}}}.\end{aligned}$$

Its Fibonacci Corollary:

Corollary 5. *For $i, j \geq 1$:*

$$\begin{aligned}\mathcal{C}_{i,j} = &\mathbf{i}^{(j\lambda+r)(j-1)}(-1)^{kr}\left\{\begin{matrix} i+j+k \\ i+1 \end{matrix}\right\}^{-1}_{(r,\lambda)}\left\{\begin{matrix} i-1 \\ j-1 \end{matrix}\right\}_{\lambda}\left(\prod_{t=0}^{j+k-2}F_{\lambda t+r}\right)^{-1} \\ &\times\left(\prod_{t=1}^{j-1}F_{\lambda t}\right)\sqrt{\left\{\begin{matrix} j+k-2 \\ k-1 \end{matrix}\right\}_{\lambda}\left\{\begin{matrix} j+k \\ j+1 \end{matrix}\right\}_{(r,\lambda)}\left(\prod_{t=0}^{k-2}F_{\lambda t+r}\right)F_{\lambda(2j+k-1)+r}}.\end{aligned}$$

From the Corollary above, we give the following examples: for $\lambda = 2$, $r = -1$,

$$\begin{aligned}\mathcal{C}_{i,j} = &\mathbf{i}^{1-j}(-1)^{k}\left\{\begin{matrix} i+j+k-1 \\ i \end{matrix}\right\}^{-1}_{(1,2)}\left\{\begin{matrix} i-1 \\ j-1 \end{matrix}\right\}_{2}\left(\prod_{t=1}^{j-1}F_{2t}\right) \\ &\times\sqrt{\left\{\begin{matrix} j+k-2 \\ k-1 \end{matrix}\right\}_{2}\left\{\begin{matrix} 2j+k-1 \\ j \end{matrix}\right\}_{(1,2)}\left(\prod_{t=1}^{2j+k-1}F_{2t-1}\right)^{-1}}\end{aligned}$$

and, for $\lambda = 2$, $r = 0$,

$$\mathcal{C}_{i,j} = (-1)^{j(j-1)}\left\{\begin{matrix} i+j+k-1 \\ i \end{matrix}\right\}^{-1}_{2}\left\{\begin{matrix} i-1 \\ j-1 \end{matrix}\right\}_{2}\sqrt{\frac{F_{2(2j+k-1)}}{F_{2j}F_{2(j+k-1)}}\left(\prod_{t=1}^{k-1}F_{2t}\right)^{-1}}.$$

2. Proofs

We compute

$$\begin{aligned}&\sum_{d} l_{md}u_{dn} \\ &= \sum_{d}\mathbf{i}^{\lambda k(d-m)}q^{\lambda\frac{k(m-d)}{2}}\frac{(q^{\lambda};q^{\lambda})_{m-1}(q^{\lambda(d+1)+r};q^{\lambda})_{d+k-1}}{(q^{\lambda};q^{\lambda})_{d-1}(q^{\lambda};q^{\lambda})_{m-d}(q^{\lambda(m+1)+r};q^{\lambda})_{d+k-1}} \\ &\times \mathbf{i}^{\lambda\frac{k}{2}(1-k)-\lambda k(n+d)+k-kr}q^{\lambda[\frac{k}{2}\left(d+n-\frac{1}{2}+\frac{k}{2}\right)-d+d^2]+\frac{k(r-1)}{2}-r+dr}(1-q)^{k} \\ &\times \frac{(q^{\lambda};q^{\lambda})_{d+k-2}(q^{\lambda};q^{\lambda})_{n-1}}{(q^{\lambda(d+k)+r};q^{\lambda})_{d-1}(q^{\lambda(n+1)+r};q^{\lambda})_{d+k-1}(q^{\lambda};q^{\lambda})_{n-d}(q^{\lambda};q^{\lambda})_{k-1}}.\end{aligned}$$

From this, we only continue with terms that depend on the summation index d:

$$\sum_d q^{\lambda(-d+d^2)+dr}\frac{(q^r;q^\lambda)_{2d+k}}{(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{m-d}(q^r;q^\lambda)_{m+d+k}} \times\frac{(q^\lambda;q^\lambda)_{d+k-2}}{(q^r;q^\lambda)_{2d+k-1}(q^r;q^\lambda)_{n+d+k}(q^\lambda;q^\lambda)_{n-d}}.$$

We set $Q := q^\lambda$ and $s = r/\lambda$:

$$\sum_d Q^{-d+d^2+ds}\frac{(Q^s;Q)_{2d+k}}{(Q;Q)_{d-1}(Q;Q)_{m-d}(Q^s;Q)_{m+d+k}} \times\frac{(Q;Q)_{d+k-2}}{(Q^s;Q)_{2d+k-1}(Q^s;Q)_{n+d+k}(Q;Q)_{n-d}}.$$

Apart from a constant factor, this is the sum that has been evaluated already in [2], when (q,r) from [2] is replaced by (Q,s).

Now we look at the inverse matrices:

$$\begin{aligned}
&\sum_{n\le d\le m} l_{m,d}l^{-1}_{d,n}\\
&=\sum_{n\le d\le m}\mathbf{i}^{\lambda k(d-m)}q^{\lambda\frac{k(m-d)}{2}}\frac{(q^\lambda;q^\lambda)_{m-1}(q^{\lambda(d+1)+r};q^\lambda)_{d+k-1}}{(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{m-d}(q^{\lambda(m+1)+r};q^\lambda)_{d+k-1}}\\
&\times\mathbf{i}^{(\lambda k+2)(n-d)}q^{\frac{\lambda}{2}(n-d)(n-k-d+1)}\frac{(q^\lambda;q^\lambda)_{d-1}(q^{\lambda(n+1)+r};q^\lambda)_{d+k-2}}{(q^\lambda;q^\lambda)_{n-1}(q^\lambda;q^\lambda)_{d-n}(q^{\lambda(d+1)+r};q^\lambda)_{d+k-2}}\\
&=\mathbf{i}^{\lambda k(n-m)}\sum_{n\le d\le m}q^{\lambda\frac{k(m-d)}{2}}\frac{(q^\lambda;q^\lambda)_{m-1}(q^{\lambda(d+1)+r};q^\lambda)_{d+k-1}}{(q^\lambda;q^\lambda)_{m-d}(q^{\lambda(m+1)+r};q^\lambda)_{d+k-1}}\\
&\times(-1)^{n-d}q^{\frac{\lambda}{2}(n-d)(n-k-d+1)}\frac{(q^{\lambda(n+1)+r};q^\lambda)_{d+k-2}}{(q^\lambda;q^\lambda)_{n-1}(q^\lambda;q^\lambda)_{d-n}(q^{\lambda(d+1)+r};q^\lambda)_{d+k-2}}.
\end{aligned}$$

We only continue with terms that depend on the summation index d:

$$\sum_{n\le d\le m}\frac{(-1)^d q^{-\lambda nd+\frac{\lambda}{2}d(d-1)}(q^{\lambda(d+1)+r};q^\lambda)_{d+k-1}(q^{\lambda(n+1)+r};q^\lambda)_{d+k-2}}{(q^\lambda;q^\lambda)_{m-d}(q^{\lambda(m+1)+r};q^\lambda)_{d+k-1}(q^\lambda;q^\lambda)_{d-n}(q^{\lambda(d+1)+r};q^\lambda)_{d+k-2}}.$$

We replace $Q := q^\lambda$, $s := r/\lambda$ and leave out irrelevant factors:

$$\sum_{n\le d\le m}\frac{(-1)^d Q^{-nd+\binom{d}{2}}(1-Q^{s+2d+k-1})(Q^s;Q)_{n+d+k-1}}{(Q;Q)_{m-d}(Q;Q)_{d-n}(Q^s;Q)_{m+d+k}}.$$

Apart from a constant factor, this is the sum that has been evaluated already in [2], when (q,r) from [2] is replaced by (Q,s).

$$\begin{aligned}
&\sum_{m\le d\le n} u_{m,d}u^{-1}_{d,n}\\
&=\sum_{m\le d\le n} \mathbf{i}^{\lambda\frac{k}{2}(1-k)-\lambda k(d+m)+k-kr} q^{\lambda[\frac{k}{2}\left(m+d-\frac12+\frac{k}{2}\right)-m+m^2]+\frac{k(r-1)}{2}-r+mr}(1-q)^k\\
&\times \frac{(q^\lambda;q^\lambda)_{m+k-2}(q^\lambda;q^\lambda)_{d-1}}{(q^{\lambda(m+k)+r};q^\lambda)_{m-1}(q^{\lambda(d+1)+r};q^\lambda)_{m+k-1}(q^\lambda;q^\lambda)_{d-m}(q^\lambda;q^\lambda)_{k-1}}\\
&\times (-1)^{\frac{\lambda k(d+n)}{2}+\frac{kr}{2}-d+\lambda\frac{k(k-1)}{4}-\frac{k}{2}+n^2}\\
&\times q^{-\lambda\frac{n(n-1)}{2}+r-\lambda\frac{k(d+n)}{2}-\frac{kr}{2}-\lambda nd+\lambda\frac{d(d+1)}{2}-\lambda\frac{k(k-1)}{4}+\frac{k}{2}-rn}\\
&\times \frac{(q^{\lambda(n+k)+r};q^\lambda)_n(q^{\lambda(d+1)+r};q^\lambda)_{n+k-2}}{(q^\lambda;q^\lambda)_{n+k-2}(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{n-d}}\frac{(q^\lambda;q^\lambda)_{k-1}}{(1-q)^k}.
\end{aligned}$$

Once again, we only write the terms that do depend on d:

$$\sum_{m\le d\le n}\frac{(-1)^d q^{-\lambda nd+\lambda\frac{d(d+1)}{2}}(q^\lambda;q^\lambda)_{d-1}}{(q^{\lambda(d+1)+r};q^\lambda)_{m+k-1}(q^\lambda;q^\lambda)_{d-m}}\frac{(q^{\lambda(d+1)+r};q^\lambda)_{n+k-2}}{(q^\lambda;q^\lambda)_{n+k-2}(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{n-d}}.$$

And again we do the usual replacement and ignore irrelevant factors:

$$\sum_{m\le d\le n}\frac{(-1)^d Q^{-nd+\frac{d(d+1)}{2}}(Q^s;Q)_{d+n+k-1}}{(Q^s;Q)_{d+m+k}(Q;Q)_{d-m}(Q;Q)_{n+k-2}(Q;Q)_{n-d}}.$$

And once again, this has been evaluated already in our previous paper.

Finally, for the Cholesky decomposition, we need to consider

$$\sum_{1\le j\le\min\{i,l\}}\mathcal{C}_{i,j}\mathcal{C}_{l,j},$$

or

$$\begin{aligned}
&\sum_{1\le j\le\min\{i,l\}}\frac{(q^\lambda;q^\lambda)_{i-1}(1-q)^{\frac{k}{2}}}{(q^{\lambda(i+1)+r};q^\lambda)_{j+k-1}(q^\lambda;q^\lambda)_{i-j}}q^{\lambda\frac{j(j-1)}{2}+\lambda\frac{ki}{2}+\lambda\frac{k^2}{8}-\lambda\frac{k}{8}-\frac{k}{4}+\frac{rj}{2}+\frac{kr}{4}-\frac{r}{2}}\\
&\times \mathbf{i}^{-\lambda\frac{k^2}{4}+\lambda\frac{k}{4}+\frac{k}{2}+\frac{3rk}{2}-\lambda ik}\sqrt{\frac{(1-q^{\lambda(2j+k-1)+r})(q^\lambda;q^\lambda)_{j+k-2}(q^{\lambda(j+1)+r};q^\lambda)_{k-1}}{(q^\lambda;q^\lambda)_{k-1}(q^\lambda;q^\lambda)_{j-1}}}\\
&\times \frac{(q^\lambda;q^\lambda)_{l-1}}{(q^{\lambda(l+1)+r};q^\lambda)_{j+k-1}(q^\lambda;q^\lambda)_{l-j}}(1-q)^{\frac{k}{2}}q^{\lambda\frac{j(j-1)}{2}+\lambda\frac{kl}{2}+\lambda\frac{k^2}{8}-\lambda\frac{k}{8}-\frac{k}{4}+\frac{rj}{2}+\frac{kr}{4}-\frac{r}{2}}\\
&\times \mathbf{i}^{-\lambda\frac{k^2}{4}+\lambda\frac{k}{4}+\frac{k}{2}+\frac{3rk}{2}-\lambda lk}\sqrt{\frac{(1-q^{\lambda(2j+k-1)+r})(q^\lambda;q^\lambda)_{j+k-2}(q^{\lambda(j+1)+r};q^\lambda)_{k-1}}{(q^\lambda;q^\lambda)_{k-1}(q^\lambda;q^\lambda)_{j-1}}}
\end{aligned}$$

We only let the terms survive that do depend on the summation index j:

$$\sum_{1\le j\le \min\{i,l\}} \frac{q^{\lambda j(j-1)+rj}}{(q^{\lambda(i+1)+r};q^{\lambda})_{j+k-1}(q^{\lambda};q^{\lambda})_{i-j}} \times \frac{(1-q^{\lambda(2j+k-1)+r})(q^{\lambda};q^{\lambda})_{j+k-2}(q^{\lambda(j+1)+r};q^{\lambda})_{k-1}}{(q^{\lambda};q^{\lambda})_{j-1}(q^{\lambda(l+1)+r};q^{\lambda})_{j+k-1}(q^{\lambda};q^{\lambda})_{l-j}}.$$

Rewriting it:

$$\sum_{1\le j\le \min\{i,l\}} \frac{Q^{j(j-1)+sj}(1-Q^{2j+k+s-1})(Q;Q)_{j+k-2}(Q^s;Q)_{j+k}}{(Q^s;Q)_{i+j+k}(Q;Q)_{i-j}(Q^s;Q)_{j+1}(Q;Q)_{j-1}(Q^s;Q)_{l+j+k}(Q;Q)_{l-j}}.$$

And this is again the sum already studied in our previous paper.

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