GRAND MOTZKIN PATHS AND $\{0,1,2\}$-TREES – A SIMPLE BIJECTION

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ABSTRACT. A well-known bijection between Motzkin paths and ordered trees with outdegree always ≤ 2, is lifted to Grand Motzkin paths (the nonnegativity is dropped) and an ordered list of an odd number of such $\{0,1,2\}$ trees. This offers an alternative to a recent paper by Rocha and Pereira Spreafico.

1. INTRODUCTION

Motzkin paths appear first in [6]. In the encyclopedia [9] they are enumerated by sequence A001006, with many references given. They consist of up-steps $U = (1,1)$, down-steps $D = (1,-1)$ and horizontal (flat) steps $F = (1,0)$. They start at the origin and must never go below the x-axis. Usually one requires the path to end on the x-axis as well, but occasionally one uses the term *Motzkin path* also for paths that end on a different level. Figure 1 shows all Motzkin paths of 4 steps (=length 4).

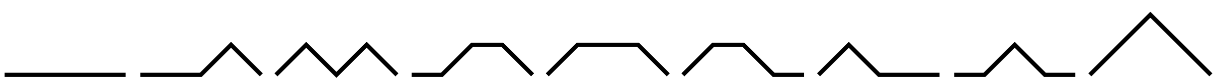

FIGURE 1. All 9 Motzkin of 4 steps (length 4).

The enumeration of Motzkin paths is done using the generating function $M = M(z)$ and a decomposition according to the first return to the x-axis, viz.

$$M = 1 + zM + z^2M^2,$$

this can be found in many books, e.g. in [5]. Solving,

$$M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}.$$

The other combinatorial structure that plays a role in this note are *ordered trees*. They are enumerated by an equation for the generating function (according to the number of nodes)

$$P = z + zP + zP^2 + zP^3 + \cdots = \frac{z}{1-P} \quad \text{and therefore} \quad P = P(z) = \frac{1 - \sqrt{1-4z}}{2}.$$

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The subclass of $\{0,1,2\}$-trees of interest only allows outdegrees $0,1,2$ and so we get again a generating function

$$Q = z + zQ + zQ^2 \quad \text{and therefore} \quad Q = Q(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z} = zM(z).$$

So there should be a bijection of $\{0,1,2\}$-trees with $n+1$ nodes ($= n$ edges) and Motzkin paths of length n; the simplest I know is from [3]:

One runs through the $\{0,1,2\}$-tree in *pre-order*; if one sees an edge for the *first* time, one translates a single edge (degree 1) into a flat step, a left edge into an up-step and a right edge into a down-step. It is easy to see that the process is reversible, which is the desired bijection.

2. Grand Motzkin paths

As a first step, we need the generating function of Motzkin paths, ending on level k, not just the usual case 0. This is a standard argument, by decomposing such a path according to the last time you visit level 0, then an up-step, and we wait until we visit level 1 for the last time, and so one.

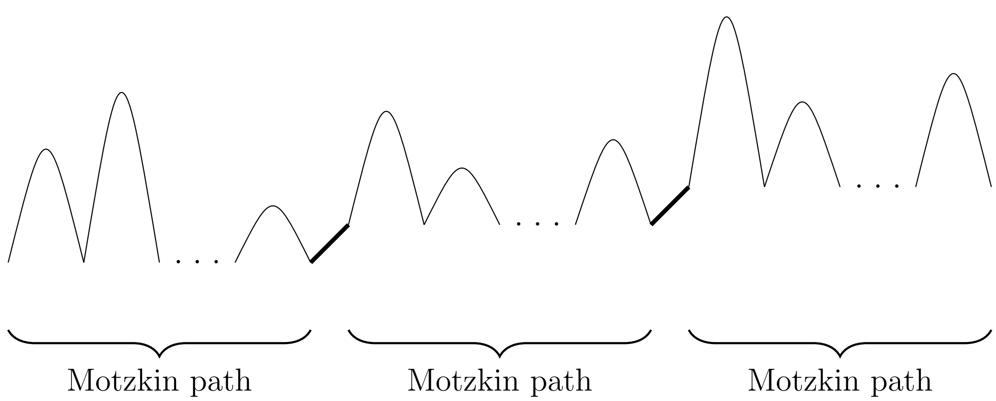

Figure 2. The decomposition of a Motzkin path that ends at level 2.

From this we find the generating function as $z^k M^{k+1}$. This is well-known.

Now we come to Grand Motzkin paths, a notation I picked up from [4]. This more general family has the same steps as Motzkin paths, returns to the x-axis at the end, but the condition that the paths must stay (weakly) above the x-axis is dropped.

Let k be the unique negative number such that the Grand Motzkin paths reaches the level $-k$ when going from left to right; for $k = 0$ they are just ordinary Motzkin paths. We consider the first such point $(a, -k)$ and the last such point $(b, -k)$; it is possible that $a = b$. Then the paths decomposes canonically into 3 parts: This decomposition produces the generating function of such paths as a product of 3 terms: The first part corresponds

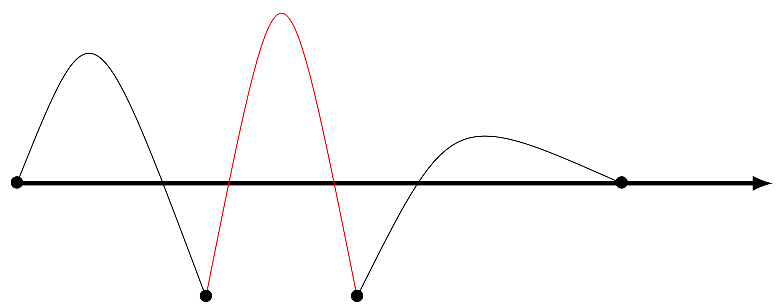

FIGURE 3. The decomposition of a Grand Motzkin path that has -3 as its minimal level.

to $z \cdot z^{k-1}M^k$ (the last step must be a down-step), the second part to M, and the third part again to $z \cdot z^{k-1}M^k$ (the first step must be an up-step). The product is then $z^{2k}M^{2k+1}$. Summing over all possible values of k, the enumeration of Grand Motzkin paths is done via the generating function

$$M + \sum_{k\ge1} z^{2k}M^{2k+1} = \sum_{k\ge0} z^{2k}M^{2k+1} = \frac{1}{\sqrt{1-2z-3z^2}}.$$

In the world of $\{0,1,2\}$-trees, we form a new super-root, with $2k+1$ successors, each of which is a $\{0,1,2\}$-tree. The generating function is then

$$z\sum_{k\ge0} Q^{2k+1} = z^2\sum_{k\ge0} z^{2k}M^{2k+1} = \frac{z^2}{\sqrt{1-2z-3z^2}}.$$

The previous bijection takes over to the new situation: if a grand Motzkin path consists of $2k+1$ ordinary Motzkin paths, then each of them corresponds to a $\{0,1,2\}$-tree as before. The extra factor z^2 stems from the fact that for $\{0,1,2\}$-trees, the edges correspond to the steps. And now, there is the super-root, which should not be counted when considering the corresponding Grand Motzkin path.

The paper [8] has a correspondence between Grand Motzkin paths and $\{0,1,2\}$-trees with a super-root with an odd number of successors. However the arguments are perhaps less direct than the present ones.

3. Trinomial coefficients and enumeration

The trinomial coefficients (notation from Comtet [1]) are given by

$$\binom{n,3}{k} = [z^k](1+z+z^2)^n.$$

These coefficients are intimately related to the Motzkin-world, as we will discuss for the reader's benefit. We use the substitution $z = \dfrac{v}{1+v+v^2}$ as we first did in [7]. Using this substitution, all generating functions become much easier, like

$$Q(z) = v, \qquad \frac{1}{\sqrt{1-2z-3z^2}} = \frac{1+v+v^2}{1-v^2}.$$

Coefficients can be extracted via contour integration, which is a variant of the Lagrange inversion formula. See [2] and [7] as illustrations of the technique. Note that when z runs around the origin in a small circle, v runs around the origin once as well, in a deformed circle. We will show two sample computations:

$$\begin{aligned}
[z^n]\frac{1}{\sqrt{1-2z-3z^2}} &= \frac{1}{2\pi i}\oint \frac{dz}{z^{n+1}}\frac{1}{\sqrt{1-2z-3z^2}}\\
&= \frac{1}{2\pi i}\oint \frac{dv(1-v^2)}{(1+v+v^2)^2}\frac{(1+v+v^2)^{n+1}}{v^{n+1}}\frac{1+v+v^2}{1-v^2}\\
&= \frac{1}{2\pi i}\oint \frac{dv}{v^{n+1}}(1+v+v^2)^n = [v^n](1+v+v^2)^n = \binom{n,3}{n};
\end{aligned}$$

and

$$\begin{aligned}
[z^n]Q^j &= \frac{1}{2\pi i}\oint \frac{dz}{z^{n+1}}v^j = \frac{1}{2\pi i}\oint \frac{dv(1-v^2)}{(1+v+v^2)^2}\frac{(1+v+v^2)^{n+1}}{v^{n+1}}v^j\\
&= \frac{1}{2\pi i}\oint \frac{dv(1-v^2)}{v^{n+1-j}}(1+v+v^2)^{n-1}\\
&= [v^{n-j}](1+v+v^2)^{n-1} - [v^{n-j-2}](1+v+v^2)^{n-1} = \binom{n-1,3}{n-j} - \binom{n-1,3}{n-j-2}.
\end{aligned}$$

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