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INFINITE 0–1-SEQUENCES WITHOUT LONG ADJACENT IDENTICAL BLOCKS

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This paper deals with sequences $a_1a_2a_3\cdots$ of symbols 0 and 1 with the property that they contain no arbitrary long blocks of the form $a_{i+1}\cdots a_{i+k}=ww$. The behaviour of this class of sequences with respect to some operations is examined. Especially the following is shown: Let be $a_i^{(0)}=a_i$, $a_i^{(n+1)}=(1/i)\sum_{k=1}^{i}a_k^{(n)}$, then there exists a sequence without arbitrary long adjacent identical blocks such that no $\lim_{k\to\infty}a_k^{(n)}$ exists. Let be $\alpha\in(0,1)$, then there exists such a sequence with $\lim_{k\to\infty}a_k^{(1)}=\alpha$. Furthermore a class of sequences appearing in computer graphics is considered.

1. Introduction

In this section first the basic definitions are given, followed by a short survey of the remaining sections.

An *alphabet* Σ is a finite nonempty set, the elements of Σ are called *symbols*. Σ^* denotes the free monoid generated by Σ. The elements of Σ^* are called *words*. The unit in Σ^* is denoted by ε. The length of a word $w\in\Sigma^*$ is denoted by $|w|$ and is 0 if $w=\varepsilon$ and n if $w=a_1\cdots a_n$, $a_i\in\Sigma$.

The *mirror image* of a word $w\in\Sigma^*$ is denoted by w^{R} and is ε if $w=\varepsilon$ and $a_n\cdots a_1$ if $w=a_1\cdots a_n$, $a_i\in\Sigma$.

An infinite sequence $a_1a_2a_3\cdots$, $a_i\in\Sigma$ is called *Σ-sequence*.

A *substitution* is a mapping $\tau:\Sigma_1^*\to\mathfrak{P}(\Sigma_2^*)$ such that the following conditions hold: $\tau(\varepsilon)=\varepsilon$ and for each $a\in\Sigma_1$ there exists $L_a\subseteq\Sigma_2^*$, such that $\tau(a_1\cdots a_n)=L_{a_1}\cdots L_{a_n}$ for all $a_1\cdots a_n\in\Sigma_1^*$. Let τ be a substitution such that for each $a\in\Sigma_1$ $\varepsilon\notin L_a$ holds. Then to each Σ_1-sequence $\omega=a_1a_2a_3\cdots$ corresponds the set $\tau(\omega)=\{w_1w_2w_3\cdots \mid w_i\in L_{a_i}\}$ of Σ_2-sequences.

Let $\omega=a_1a_2a_3\cdots$ be a Σ-sequence, $a\in\Sigma$, $k\in\mathbf{N}$, then $n_a^{(\omega)}(k)$ denotes the number of symbols a in $a_1\cdots a_k$.

A word $x\in\Sigma^*$ is called *subword* of a word $w\in\Sigma^*$ (of a Σ-sequence ω), if there are words y, $z\in\Sigma^*$ (a word $y\in\Sigma^*$ and a Σ-sequence η), such that $w=yxz$ $(\omega=y\,x\,n)$.

For $\Sigma=\{0,1\}$, $\tau(0)=1$, $\tau(1)=0$, $\tau(w)$ $(\tau(\omega))$ are abbreviated by $\bar{w}$ $(\bar{\omega})$.

A $\{0,1\}$-sequence ω has arbitrary long adjacent identical blocks (*is of unbounded repetition*) provided that for all $n\in\mathbf{N}$ there exists a subword ww of ω where $|w|\geq n$. A sequence not of this type is called sequence of *bounded repetition*.

The existence of sequences of unbounded repetition is evident. The second section contains a historical remark concerning the existence of sequences of bounded repetition; a special one is discussed in detail in Section 3, these examinations bring up some interesting arithmetical identities.

The *relative frequencies* of symbols 1 in sequences with bounded repetition are examined in Section 4.

Section 5 contains some results about operations on sequences of (un-) bounded repetition.

In the last section a class of sequences with unbounded repetition is related to a problem appearing in computer graphics.

2. Historical remark

It is well-known (Thue [12], Arshon [1], Hedlund and Morse [6]), that there are $\{0,1,2\}$-sequences containing no subword of the form ww. Such $\{0,1,2\}$-sequences can be used in order to construct sequences of bounded repetition.

Entringer, Jackson and Schatz [4] have shown that there are $\{0,1\}$-sequences having only subwords ww with $|w|\leq 2$ and that this constant cannot be improved. The construction is based on a $\{0,1,2\}$-sequence containing no subword of the form ww and the substitution $\tau(0)=1010$, $\tau(1)=1100$, $\tau(2)=0111$.

It is remarked that the substitution $\tau(0)=0000$, $\tau(1)=0101$, $\tau(2)=1111$ is also possible.

A further sequence of bounded repetition can be constructed as in Section 3: The sequence $0000\cdots$ is written down. Between every two symbols a gap is left. Now the sequence $1111\cdots$ is filled in the gaps, where gaps of odd index are left free. In the remaining (infinitely many) gaps the sequence $000\cdots$ is written, where again gaps of odd index are left free. This process (inserting 0's and 1's) is repeated ad infinitum. The nth element of this sequence can be obtained in the following way: if $n=2^{k+1}i+2^k$, then $a_n\equiv k \pmod 2$.

3. A special sequence with bounded repetition

Let be $\omega=a_1a_2a_3\cdots$, where $a_n\in\{0,1\}$, $a_n\equiv i \pmod 2$ if $n=2^{k+1}i+2^k$. Since each $n\in\mathbf{N}$ can be uniquely written as $n=2^{k+1}i+2^k$, ω is well defined. (If the binary representation of n is $w\sigma 10\cdots 0$, $\sigma\in\{0,1\}$, then $a_n=\sigma$). ω can be defined as follows (see Jacobs and Keane [7]):

The sequence $0101\cdots$ is written down, leaving a gap between every two symbols:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}
0		1		0		1		0		1		0	

Now the sequence $0101\cdots$ is filled in the gaps, leaving free every second gap:

$$\begin{array}{cccccccccccccc}
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\
0 & & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & \\
& 0 & & & & 1 & & & & 0 & & & & 1
\end{array}$$

The remaining gaps are again filled by the sequence $0101\cdots$, leaving free every second gap:

$$\begin{array}{cccccccccccccc}
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\
0 & & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & \\
& 0 & & & & 1 & & & & 0 & & & & 1 \\
& & & 0 & & & & & & & & 1 & &
\end{array}$$

This process is repeated ad infinitum.

Theorem 3.1. *ω is a sequence of bounded repetition.*

Proof. It will be shown by induction on $n \geqslant 6$, that ω contains no word of the form xx with $|x| = n$. (A separate discussion of the cases $n = 6$, 7, 8, 9 and 10 is necessary.)

(i) $n = 6$. In a set of 6 consecutive natural numbers there is always a k with $k \equiv 1 \pmod 8$ or $k \equiv 5 \pmod 8$. The binary representation of k ends in both cases with 01, therefore $a_k = 0$. Thus the binary representation of $k+6$ ends with 11, and so $a_{k+6} = 1$. It follows that two consecutive words of length 6 in ω differ at least at one position.

Similar arguments are used in the following cases:

(ii) $n = 7$. In a set of 7 consecutive natural numbers there is always a k with $k \equiv 3 \pmod 8$ or $k \equiv 6 \pmod 8$. Therefore $a_k = 1$, but $a_{k+7} = 0$.

(iii) $n = 8$. In a set of 8 consecutive natural numbers there is always a k with $k \equiv 4 \pmod{16}$ or $k \equiv 12 \pmod{16}$. In the first case $a_k = 0$ and $a_{k+8} = 1$, in the second case $a_k = 1$ and $a_{k+8} = 0$.

(iv) $n = 9$. For $k \equiv 5, 13 \pmod{16}$ $a_k = 0$ and $a_{k+9} = 1$.

(v) $n = 10$. For $k \equiv 4, 13 \pmod{16}$ $a_k = 0$ and $a_{k+10} = 1$.

(vi) Since $a_2 a_4 a_6 \cdots = \omega = a_1 a_2 a_3 \cdots$, ω contains to each subword xx, where $|x| = 2k$ already a subword yy, where $|y| = k$ (yy is obtained by erasing all symbols of xx with odd index). Therefore the statement holds for even n.

(vii) Let be $n \geqslant 11$ an odd number and $a_{i+1} \cdots a_{i+n} a_{i+n+1} \cdots a_{i+2n}$ a subword of ω of the form xx. Let $k \in \{i+1, i+2\}$ be odd. Then

$$a_k a_{k+1} \cdots a_{k+8} = \sigma a_{k+1} \bar{\sigma} a_{k+3} \sigma a_{k+5} \bar{\sigma} a_{k+7} \sigma = \alpha.$$

Since $k+n+1$ is odd, in the same way it can be concluded that

$$a_{k+n} a_{k+n+1} \cdots a_{k+n+8} = a_{k+n} \tau a_{k+n+2} \bar{\tau} a_{k+n+4} \tau a_{k+n+6} \bar{\tau} a_{k+n+8} = \beta$$

and $\alpha = \beta$ must hold. Therefore $\alpha = \sigma\tau\bar{\sigma}\bar{\tau}\sigma\tau\bar{\sigma}\bar{\tau}\sigma = \beta$. Without loss of generality let be $\sigma=\tau$ (otherwise σ is replaced by τ and τ by $\bar{\sigma}$). $a_{i+1}\cdots a_{i+n}$ contains a subword

$$\gamma=\sigma\sigma\bar{\sigma}\bar{\sigma}\sigma\sigma\bar{\sigma}\bar{\sigma}=a_{j+1}\cdots a_{j+8}.$$

Then there is a unique r, $j+1\leqslant r\leqslant j+4$, such that $r\equiv 2 \pmod 8$ or $r\equiv 6 \pmod 8$. Like in (i)–(v) it can be concluded, that $a_r\neq a_{r+4}$, which is impossible because of the form of γ. (This reasoning also excludes, that ω contains a subword xx, where $|x|=4$.)

In Theorem 3.3 it will be shown, that ω can be defined recursively (similar to Hedlund and Morse [6]).

The following lemma will then be used:

Lemma 3.2. $a_1\cdots a_{2^n-1}=\overline{a_{2^n+1}\cdots a_{2^{n+1}-1}}^R$ *for all* $n\in\mathbf{N}$.

Proof. Let be $1\leqslant i\leqslant 2^n-1$ and $w\sigma 10^k$ the binary representation of i. Then $1\bar{w}\bar{\sigma}10^k$ is the binary representation of $2^{n+1}-i$ and therefore $a_i=\overline{a_{2^{n+1}-i}}$.

Theorem 3.3. *Let be* α_n, β_n, $n\geqslant 1$ *recursively defined as follows:*

$$\alpha_1=0,\quad \beta_1=1,\qquad \alpha_{n+1}=\alpha_n 0\beta_n,\quad \beta_{n+1}=\alpha_n 1\beta_n,\quad n\geqslant 1.$$

Then $a_1\cdots a_{2^n-1}=\alpha_n$ *for all* $n\in\mathbf{N}$.

Proof. First, by induction on n, it is shown that $\alpha_n=\bar{\beta}_n^R$:

(i) $\alpha_1=0=\bar{1}^R=\bar{\beta}_1^R$,

(ii) $\alpha_{n+1}=\alpha_n 0\beta_n=\bar{\beta}_n^R\bar{1}^R\bar{\alpha}_n^R=\overline{\alpha_n 1\beta_n}^R=\bar{\beta}_{n+1}^R$.

Now the statement of the theorem is proved by induction on n:

(i) $a_1=0=\alpha_1$,

(ii) $a_1\cdots a_{2^n-1}a_{2^n}a_{2^n+1}\cdots a_{2^{n+1}-1}=a_1\cdots a_{2^n-1}a_{2^n}\overline{a_1\cdots a_{2^n-1}}^R$
$=\alpha_n 0\bar{\alpha}_n^R=\alpha_n 0\beta_n=\alpha_{n+1}$.

In the rest of this section the numbers $n_1^{(\omega)}(k)$ and $\lim_{k\to\infty} n_1^{(\omega)}(k)/k$ are examined. (Since there is no danger of confusion, $n_1(k)$ will be written instead of $n_1^{(\omega)}(k)$.)

Definition 3.4. Let be $k\in\mathbf{N}_0$. The *variation* $v(k)$ of k is defined recursively as follows: $v(0)=0$, $v(2j+i)=v(j)+\delta$, where $i,\delta\in\{0,1\}$, $\delta\equiv i+j \pmod 2$.

Roughly spoken, $v(k)$ denotes the number of changes of consecutive digits in the binary representation of k, where the leftmost digit 1 counts as a change.

The following lemma shows a property of $v(k)$ which is used in the sequel.

Lemma 3.5. *Let be* $2^n\leqslant k<2^{n+1}$. *Then* $v(k)=v(2^{n+1}-k-1)+1$.

Proof. By induction on n:

(i) If $n=0$, then only $k=1$ is possible and $v(1)=1=v(0)+1$.

(ii) Let be $n\geqslant 1$ and $2^n\leqslant k<2^{n+1}$. Then $k=2j+1$, $i\in\{0,1\}$ and $2^{n-1}\leqslant j<2^n$. Let be $\delta,\delta'\in\{0,1\}$, $\delta\equiv i+j \pmod 2$, $\delta'\equiv 2^n-j-i \pmod 2$. Then $\delta\equiv\delta' \pmod 2$ and

$$\begin{aligned}v(k)&=v(2j+i)=v(j)+\delta=v(2^n-j-1)+1+\delta\\&=v(2^n-j-1)+\delta'+1=v(2(2^n-j-1)+1-i)+1\\&=v(2^{n+1}-2j-i-1)+1=v(2^{n+1}-k-1)+1.\end{aligned}$$

Now the numbers $n_1(k)$ and $v(k)$ can be related:

Theorem 3.6. $n_1(k)=\frac{1}{2}(k-v(k))$.

Proof. Let be $2^n\leqslant k<2^{n+1}$. The statement is proved by induction on n:

(i) If $n=0$, then $k=1$ and $n_1(1)=0=\frac{1}{2}(1-v(1))$.

(ii) Let be $n\geqslant 1$. The number of symbols 1 in $a_1\cdots a_{2^n-1}a_{2^n}\cdots a_k$ can be determined in the following manner (it should be remembered that $a_{2^n}=0$): In $a_1\cdots a_{2^n-1}$ occur exactly $n_1(2^n-1)=\frac{1}{2}(2^n-1-v(2^n-1))$ symbols 1. To this number the number m of symbols 1 in $a_{2^n+1}\cdots a_{2^{n+1}-1}$ is added, and the number m' of symbols 1 in $a_{k+1}\cdots a_{2^{n+1}-1}$ is subtracted. Lemma 3.2 implies that

$$m=n_0(2^n-1)=2^n-1-n_1(2^n-1)$$

and

$$m'=n_0(2^{n+1}-k-1)=2^{n+1}-k-1-n_1(2^{n+1}-k-1).$$

Therefore

$$\begin{aligned}n_1(k)&=n_1(2^n-1)+m-m'\\&=n_1(2^n-1)+2^n-1-n_1(2^n-1)-2^{n+1}+k+1+n_1(2^{n+1}-k-1)\\&=-2^n+k+\tfrac{1}{2}(2^{n+1}-k-1-v(2^{n+1}-k-1))\\&=\tfrac{1}{2}(k-1-(v(k)-1))\\&=\tfrac{1}{2}(k-v(k)).\end{aligned}$$

Now it can be shown that the sequence $n_1(k)/k$ of the relative frequencies of symbols 1 in ω converges:

Theorem 3.7. $\lim_{k\to\infty} n_1(k)/k=\frac{1}{2}$.

Proof. Since $0\leqslant v(k)\leqslant 1+\operatorname{ld} k$ always hold, it follows that

$$\tfrac{1}{2}(k-1-\operatorname{ld} k)\leqslant n_1(k)\leqslant\tfrac{1}{2}k.$$

Therefore

$$\frac{1}{2}\left(1-\frac{1+\operatorname{ld} k}{k}\right)\leqslant\frac{n_1(k)}{k}\leqslant\tfrac{1}{2}.$$

Since $\lim_{k\to\infty}(1+\operatorname{ld} k)/k=0$, the proof is finished.

Another way to compute $n_1(k)$ shows

Theorem 3.8. $n_1(k)=\sum_{i\geqslant 0}[(k+2^i)/2^{i+2}]$.

Proof. First it should be noted that $a_s=1$ if and only if there exists a number i, such that $s\equiv 3\cdot 2^i \pmod{2^{i+2}}$. (The binary representation of s must be of the form $w110^i$.) Let i be fixed. Then there are $[(k-3\cdot 2^i)/2^{i+2}]+1$ numbers s, such that $s\leqslant k$ and $s\equiv 3\cdot 2^i \pmod{2^{i+2}}$. (The following fact was used: For given n, r, m, $0\leqslant r<m$, there are exactly $[(n-r)/m]+1$ numbers t, such that $0\leqslant t\leqslant n$ and $t\equiv r \pmod{m}$.)

Furthermore

$$\left[\frac{k-3\cdot 2^i}{2^{i+2}}\right]+1=\left[\frac{k-3\cdot 2^i+2^{i+2}}{2^{i+2}}\right]=\left[\frac{k+2^i}{2^{i+2}}\right].$$

A summation over i completes the proof.

Using Lemma 3.6 and Theorem 3.8 an interesting identity can be proved:

Corollary 3.9. $\sum_{i\geqslant 0}[(k+2^i)/2^{i+2}]=\frac{1}{2}(k-v(k))$.

In a similar way an other identity can be easily proved.

Theorem 3.10. $\sum_{i\geqslant 0}[(k+2^i)/2^{i+1}]=k$.

Proof. Let $\omega'=b_1b_2b_3\cdots$ be defined as ω, but using $1111\cdots$ instead of $0101\cdots$. Then clearly $n_1^{(\omega')}(k)=k$ holds for all k.

$n_1^{(\omega')}(k)$ can be determined as in the proof of Theorem 3.8: $b_s=1$ if and only if there exists a number i, such that $s\equiv 2^i \pmod{2^{i+1}}$. (The binary representation of s must be of the form $w10^i$.)

Let i be fixed. Then there are $[(k-2^i)/2^{i+1}]+1$ numbers s such that $s\leqslant k$ and $s\equiv 2^i \pmod{2^{i+1}}$. Since

$$\left[\frac{k-2^i}{2^{i+1}}\right]+1=\left[\frac{k+2^i}{2^{i+1}}\right],$$

a summation over i completes the proof.

4. Some properties of sequences with (un-)bounded repetition

In the sequel it will be shown, that for each $\alpha \in (0, 1)$ there exists a sequence with bounded repetition $\omega = a_1 a_2 a_3 \cdots$, such that $\lim_{k\to\infty} n_1^{(\omega)}(k)/k = \alpha$. To obtain this, it is necessary to make some preparations.

In the following τ denotes the substitution $\tau(0) = \{00, 01\}$, $\tau(1) = \{11\}$.

Lemma 4.1. *Let ω be a sequence with bounded repetition. Then $\tau(\omega)$ contains only sequences with bounded repetition.*

Proof. Assume k to be a number such that ω does not contain a subword ww, where $|w| \geq k$.

Assume that there is a sequence with unbounded repetition $\eta \in \tau(\omega)$. Then η contains a subword $a_{i+1} \cdots a_{i+m} a_{i+m+1} \cdots a_{i+2m}$, where $m \geq 2k$.

It is necessary to distinguish the following cases:

(i) $i \equiv 0 \pmod 2$ and $m \equiv 0 \pmod 2$. Then there is a subword ww in ω, $|w| \geq k$, corresponding by τ to the subword $a_{i+1} \cdots a_{i+2m}$.

(ii) $i \equiv 0 \pmod 2$ and $m \equiv 1 \pmod 2$. If $a_{i+m} a_{i+m+1} = 0x$, then $a_{i+2m} = 0$ and therefore $a_{i+2m-1} = 0$, and therefore $a_{i+m-1} = 0$, etc. Because of this ω contains a subword $0^r 0^r$, where $r \geq k$. If $a_{i+m} a_{i+m+1} = 11$, then $a_{i+1} = 1$, and therefore $a_{i+2} = 1$, and therefore $a_{i+m+2} = 1$, etc. Because of this ω contains a subword $1^r 1^r$, where $r \geq k$.

(iii) $i \equiv 1 \pmod 2$ and $m \equiv 0 \pmod 2$. In this case ω contains a subword of the form $\sigma_1 w \sigma_2 w \sigma_3$, where $|w| \geq k-1$. From this it follows that $\sigma_1 \neq \sigma_2$ and $\sigma_2 \neq \sigma_3$ must hold. If $\sigma_2 = 0$, then $\sigma_3 = 1$ and therefore $a_{i+m} = 0$ and $a_{i+2m} = 1$; this is impossible. If $\sigma_2 = 1$, then $\sigma_3 = 0$ and therefore $a_{i+m} = 1$ and $a_{i+2m} = 0$; this is also impossible.

(iv) $i \equiv 1 \pmod 2$ and $m \equiv 1 \pmod 2$. In this case ω contains a subword of the form $\sigma_1 w \sigma_2 \sigma_3 w \sigma_4$, where $|w| \geq k-1$. Therefore $\sigma_1 \neq \sigma_3$ or $\sigma_2 \neq \sigma_4$ must hold. Without loss of generality one can assume that $\sigma_1 \neq \sigma_3$. If $\sigma_1 = 0$, then $\sigma_3 = 1$ and therefore $a_{i+m+1} = a_{i+m+2} = 1$, etc. Because of this ω contains a subword $0^r 0^r$, where $r \geq k$. The case $\sigma_1 = 1, \sigma_3 = 0$ can be discussed with similar arguments.

Lemma 4.2. *Let ω be a $\{0, 1\}$-sequence and $\lim_{k\to\infty} n_1^{(\omega)}(k)/k = \alpha$. Then for each $\beta \in [\alpha, \frac{1}{2}(\alpha+1)]$ there is a $\eta \in \tau(\omega)$, such that $\lim_{k\to\infty} n_1^{(\eta)}(k)/k = \beta$.*

Proof. If $\beta = \alpha$, η is obtained from ω by replacing each 0 by 00 and each 1 by 11.

Now it is assumed, that $\beta > \alpha$. Then it exists a k_0, such that $n_1^{(\omega)}(k)/k \leq \beta$ holds for all $k \geq k_0$. All symbols 0 in ω are replaced by 00 until k_0 is reached. Then all symbols 0 are replaced by 01, until a minimal k_1 is found, such that $n_1^{(\eta)}(2k_1)/2k_1 \geq \beta$. (This is possible: if all but finitely many symbols in ω are replaced by 01, then the sequence of relative frequencies of this new sequence converges to $\frac{1}{2}(\alpha+1)$.) Beginning with index k_1+1 all symbols 0 are again replaced by 00, until a minimal k_2 is found, such that $n_1^{(\eta)}(2k_2)/2k_2 \leq \beta$. This process is repeated. Clearly, the so constructed sequence η has the desired property. (Compare this construction with Knopp [8; p. 329].)

Lemma 4.3. *Let ω be a $\{0, 1\}$-sequence and $\lim_{k\to\infty} n_1^{(\omega)}(k)/k = \alpha$. Then for each $\beta \in [\alpha, 1)$ there exists a n and a $\eta \in \tau^n(\omega)$, such that $\lim_{k\to\infty} n_1^{(\eta)}(k)/k = \beta$.*

Proof. Since the sequence $(\alpha+2^k-1)/2^k$ increases strictly monotonously and converges to 1, there is a unique n, such that

$$\frac{\alpha+2^n-1}{2^n} \leq \beta < \frac{\alpha+2^{n+1}-1}{2^{n+1}} = \left(\frac{\alpha+2^n-1}{2^n}+1\right)\Big/2.$$

Then there is a $\eta' \in \tau^n(\omega)$, such that

$$\lim_{k\to\infty} \frac{n_1^{(\eta')}(k)}{k} = \frac{\alpha+2^n-1}{2^n}.$$

Then, because of Lemma 4.2, there is a $\eta \in \tau(\eta')$ $(\subseteq \tau^{n+1}(\omega))$, such that $\lim_{k\to\infty} n_1^{(\eta)}(k)/k = \beta$.

Theorem 4.4. *For each $\beta \in [\frac{1}{2}, 1)$ there is a sequence with bounded repetition η, such that $\lim_{k\to\infty} n_1^{(\eta)}(k)/k = \beta$.*

Proof. Let ω be the sequence of Section 3. Then the statement is evident applying Lemma 4.3 to ω.

Theorem 4.5. *For each $\beta \in (0, 1)$ there is a sequence with bounded repetition η, such that $\lim_{k\to\infty} n_1^{(\eta)}(k)/k = \beta$.*

Proof. The statement must be proved only for $\beta \in (0, \frac{1}{2}]$. Let ω be a sequence with bounded repetition and $\lim_{k\to\infty} n_1^{(\omega)}(k)/k = 1-\beta$. Then for $\eta = \bar{\omega}$ the statement is true.

Corollary 4.6. *The set of sequences with bounded repetition has cardinality $2^{\aleph_0}$.*

This statement can be seen also in that way: From the work of Kakutani (cf. Gottschalk and Hedlund [5; p. 109]) there are $2^{\aleph_0}$ square-free $\{0, 1, 2\}$-sequences. This can be found also in Bean, Ehrenfeucht and McNulty [2]. Then a substitution as in Section 2 gives the result.

By $\omega = a_1a_2a_3 \cdots \mapsto \Phi(\omega) = \sum a_i/2^i$, each $\{0, 1\}$-sequence can be associated with a real number in [0, 1]. Each real number which corresponds to a sequence with bounded repetition is non-normal in accordance to Borel [3]. (See Niven

[10].) Since the set of non-normal numbers is of measure 0, the following theorem holds:

Theorem 4.7. *Let M be the set of all sequences with bounded repetition. Then $\Phi(M)$ is of measure* 0.

Finally, it is shown, that there are sequences with bounded repetition, for which the sequences of the "nth averages" do not converge.

Definition 4.8. Let $\omega = a_1 a_2 a_3 \cdots$ be a $\{0, 1\}$-sequence and the sequences $a_1^{(n)}$, $a_2^{(n)}$, $a_3^{(n)}, \ldots$ of the nth averages ($n \geqslant 0$) defined by:

$$a_i^{(0)} = a_i, \qquad a_i^{(n+1)} = \frac{1}{i} \sum_{k=1}^{i} a_k^{(n)}.$$

Then $a_k^{(1)} = n_1(k)/k$.

Theorem 4.9. *There is a sequence with bounded repetition η, such that no $\lim_{k\to\infty} a_k^{(n)}$ exists.*

Proof. Let ω be the sequence of Section 3. η will be constructed by applying the substitution τ to ω step by step. For this purpose let α, β be so that $\frac{1}{2} < \alpha < \beta < \frac{3}{4}$.

First step: Symbols 0 are replaced by 01 until $a_{n_1}^{(1)} \geqslant \beta$. Then symbols 0 are replaced by 00 until $a_{m_1}^{(1)} \leqslant \alpha$.

kth step: Symbols 0 are replaced by 01 until $a_{n_k}^{(1)} \geqslant \beta$ and $a_{n_k}^{(2)} \geqslant \beta$ and $\cdots$ and $a_{n_k}^{(k)} \geqslant \beta$. Then symbols 0 are replaced by 00 until $a_{m_k}^{(1)} \leqslant \alpha$ and $\cdots$ and $a_{m_k}^{(k)} \leqslant \alpha$.

For each k the sequence $a_1^{(k)}, a_2^{(k)}, a_3^{(k)}, \ldots$ contains infinitely many numbers $\leqslant \alpha$ and $\geqslant \beta$ and therefore it does not converge.

5. Operations on sequences with (un-) bounded repetition

The behaviour of sequences with (un-) bounded repetition is examined for the following operations: changes of finite character, mixing, addition mod 2.

Lemma 5.1. *Let ω be a $\{0, 1\}$-sequence and $\sigma \in \{0, 1\}$. Then ω is a sequence with bounded repetition if and only if $\sigma\omega$ is a sequence with bounded repetition.*

Proof. If $\sigma\omega$ is a sequence with bounded repetition, then there is a k, such that $\sigma\omega$ contains no subword ww, where $|w| \geqslant k$. Then ω does not contain such a subword and is therefore a sequence with bounded repetition.

Let conversely ω be a sequence with bounded repetition. Then there is a k, such that ω contains no subword ww, where $|w| \geqslant k$. Assuming $\sigma\omega$ to be a sequence with unbounded repetition yields $\sigma\omega = xx\eta_1 = yy\eta_2$, where $|x| \geq k$ and $|y| \geq 2|x|$. Then $y = uxxv$, and ω would contain xx as a subword.

Theorem 5.2. *Let ω be a $\{0,1\}$-sequence and $x, y \in \{0,1\}^*$. Then $x\omega$ is a sequence with bounded repetition if and only if $y\omega$ is a sequence with bounded repetition.*

Proof. It follows from Lemma 5.1 by induction, that $x\omega$ is a sequence with bounded repetition if and only if ω is a sequence with bounded repetition. By a similar argument it can be concluded, that ω is a sequence with bounded repetition if and only if $y\omega$ is a sequence with bounded repetition.

Remark. Theorem 5.2 shows, that by changes of finite character (deleting and inserting of finitely many symbols) of sequences with bounded repetition again sequences with bounded repetition are obtained.

Let ω_2 be obtained from ω_1 by changes of finite character, and k_i $(i = 1, 2)$ minimal, such that ω_i contains no subword ww, $|w| \geq k_i$. Then k_1 and k_2 can be quite different.

Definition 5.3. For $\{0,1\}$-sequences $\omega = a_1a_2a_3\cdots$ and $\eta = b_1b_2b_3\cdots$ let

$$\omega\Box\eta = a_1b_1a_2b_2a_3b_3\cdots.$$

Theorem 5.4. *The sequences with unbounded repetition are not closed under $\Box$.*

Proof. Let ω be a sequence with bounded repetition, $\tau_1(0) = 00$, $\tau_1(1) = 11$, $\tau_2(0) = 01$, $\tau_2(1) = 11$. Then according to Lemma 4.1 $\tau_1(\omega) = a_1a_2a_3\cdots$ and $\tau_2(\omega) = b_1b_2b_3\cdots$ are sequences with bounded repetition. Let the sequences $\eta_1 = a'_1a'_2a'_3\cdots$ and $\eta_2 = b'_1b'_2b'_3\cdots$ be constructed as follows:

For all $n \geq 0$ let

$$a'_{2^n}\cdots a'_{2^{n+1}-1} = \begin{cases} 0^{2^n} & \text{if } n \text{ is even,} \\ a_{2^n}\cdots a_{2^{n+1}-1} & \text{if } n \text{ is odd,} \end{cases}$$

$$b'_{2^n}\cdots b'_{2^{n+1}-1} = \begin{cases} b_{2^n}\cdots b_{2^{n+1}-1} & \text{if } n \text{ is even,} \\ 1^{2^n} & \text{if } n \text{ is odd.} \end{cases}$$

Since η_1 and η_2 contain subwords 0^k0^k and 1^k1^k for infinitely many k, they are sequences with unbounded repetition.

Now it is shown that $\eta_1 \Box \eta_2$ is a sequence with bounded repetition. Assuming the contrary the following cases are possible:

(i) $\quad a'_{r+1}b'_{r+1}\cdots a'_{r+n}b'_{r+n} = a'_{r+n+1}b'_{r+n+1}\cdots a'_{r+2n}b'_{r+2n}$.

Then $a'_{r+1}\cdots a'_{r+n} = a'_{r+n+1}\cdots a'_{r+2n}$ and $b'_{r+1}\cdots b'_{r+n} = b'_{r+n+1}\cdots b'_{r+2n}$, which is possible only for finitely many n.

(ii) $b'_{r+1}a'_{r+2}\cdots b'_{r+n}a'_{r+n+1}=b'_{r+n+1}a'_{r+n+2}\cdots b'_{r+2n}a'_{r+2n+1}$

is discussed similar to (i).

(iii) $a'_{r+1}b'_{r+1}\cdots b'_{r+n}a'_{r+n+1}=b'_{r+n+1}a'_{r+n+2}\cdots a'_{r+2n+1}b'_{r+2n+1}$.

For a sufficiently large n there is a i ($1\leq i\leq n$), such that $a'_{r+i}a'_{r+i+1}=00$ and therefore $b'_{r+i+n}b'_{r+i+n+1}=00$, which is excluded by the construction of η_2.

(iv) $b'_{r+1}a'_{r+2}\cdots a'_{r+n+1}b'_{r+n+1}=a'_{r+n+2}b'_{r+n+2}\cdots b'_{r+2n+1}a'_{r+2n+2}$

is discussed similar to (iii).

If ω and η are sequences with bounded repetition, then it is quite possible, that $\omega\,\square\,\eta$ is a sequence with bounded repetition. (An example: $\omega\,\square\,\omega=\tau_1(\omega)$, τ_1 from Theorem 5.4.) It could not be found out, whether or not this holds in general. however the following can be shown:

Theorem 5.5. *For each sequence with bounded repetition ω there is a $\{0,1\}$-sequence η, such that $\omega\,\square\,\eta$ is a sequence with unbounded repetition.*

Proof. Let $\omega=a_1a_2a_3\cdots$ be a sequence with bounded repetition and $\eta=b_1b_2b_3\cdots$ be constructed as follows: for all $n\geq 0$ let be b_{2^n} be anyhow and

$$b_{2^n+1}\cdots b_{2^{n+1}-1}=a_{2^n+2^{n-1}+1}\cdots a_{2^{n+1}-1}a_{2^n+1}\cdots a_{2^n+2^{n-1}}.$$

Then $\omega\,\square\,\eta$ contains for all n the subword

$$a_{2^n+1}a_{2^n+2^{n-1}+1}\cdots a_{2^{n+1}-1}a_{2^n+2^{n-1}}a_{2^n+1}a_{2^n+2^{n-1}+1}\cdots a_{2^{n+1}-1}a_{2^n-2^{n-1}}$$

of the form ww, the length of which is $2(2^n-1)$.

Interpreting 0 and 1 as the elements of $GF(2)$, and defining the addition of $\{0,1\}$-sequences elementwise, it can be shown that neither the sequences of unbounded repetition nor the sequences with bounded repetition are closed under addition.

Theorem 5.6. *There are sequences of bounded repetition $\omega_1,\omega_2,\omega_3$ and sequences with unbounded repetition η_1,η_2,η_3, such that*

(i) $\omega_1+\omega_2=\omega_3$,	(iv) $\omega_1+\eta_2=\eta_3$,
(ii) $\omega_1+\omega_1=\eta_1$,	(v) $\eta_2+\eta_3=\omega_1$,
(iii) $\omega_1+\eta_1=\omega_1$,	(vi) $\eta_1+\eta_1=\eta_1$.

Proof. Let be $\eta_1=0000\cdots$. Then (ii), (iii) and (vi) are true. Let ω be a sequence with bounded repetition, $\omega_1=\tau_2(\omega)$, $\omega_2=1\omega_1$ and $\omega_3=\tau_1(\bar{\omega})$ (τ_i from Theorem 5.4). Since $\tau_2(\omega)+1\tau_2(\omega)=\tau_1(\bar{\omega})$, (i) holds.

Let be $\omega_1 = a_1a_2a_3\cdots$. Then $\eta_2 = b_1b_2b_3\cdots$ and $\eta_3 = c_1c_2c_3\cdots$ can be constructed in the following way:

$$b_{2^n}\cdots b_{2^{n+1}-1} = \begin{cases} 0^{2^n} & \text{if } n \text{ is even,} \\ a_{2^n}\cdots a_{2^{n+1}-1} & \text{if } n \text{ is odd,} \end{cases}$$

$$c_{2^n}\cdots c_{2^{n+1}-1} = \begin{cases} 0^{2^n} & \text{if } n \text{ is odd,} \\ a_{2^n}\cdots a_{2^{n+1}-1} & \text{if } n \text{ is even.} \end{cases}$$

Then (iv) and (v) are true.

6. A class of sequences with unbounded repetition and a problem in computer graphics

Given a line $y = \alpha x$, $0 \leq \alpha \leq 1$, which is to be drawn approximately for $x \geq 0$ by an 8-directional-plotter in a way that the errors measured along the ordinate are minimal, the points $(n, [\alpha n + \frac{1}{2}])$ must be connected. The numbers $b_n = [\alpha n + \frac{1}{2}] - [\alpha(n-1) + \frac{1}{2}]$ are in $\{0, 1\}$ and correspond to the instructions for the plotter (in the $n+1$-th step a line between the points (n, m) and $(n+1, m+b_n)$ is drawn). (Cf. Prodinger et al. [11].)

The sequence $b_1b_2b_3\cdots$ is periodic with period q, if and only if $\alpha = p/q$ is rational. For irrational α, the sequence $b_1b_2b_3\cdots$ is not periodic, but the following theorem holds:

Theorem 6.1. *For each $\alpha \in [0, 1]$ the sequence $b_1b_2b_3\cdots$ is a sequence with unbounded repetition.*

Proof. If α is rational, the statement is evident. Let α be irrational. The sequence $\alpha k \pmod 1$ is dense in $[0, 1)$ (Kuipers and Niederreiter [9; p. 23]).

Let be $n' > 0$. It will be shown that there is a $n \geq n'$, such that the sequence $b_1b_2b_3\cdots$ contains a subword ww, where $|w| = n$. Since $\alpha k \pmod 1$ is dense in $[0, 1)$, there is a minimal $n \geq n'$, such that

$$0 < \alpha n \ (\mathrm{mod}\ 1) < \lim_{1 \leq i \leq n'} (\alpha i (\mathrm{mod}\ 1)).$$

Let be $\varepsilon = \alpha n \pmod 1$,

$$\delta_1 = \min_{\substack{1 \leq i \leq n \\ \alpha i < 1/2}} (\tfrac{1}{2} - (\alpha i (\mathrm{mod}\ 1))), \qquad \delta_2 = \min_{\substack{1 \leq i \leq n \\ \alpha i > 1/2}} (\alpha i (\mathrm{mod}\ 1) - \tfrac{1}{2}).$$

Since n was chosen minimal, $\varepsilon < \delta_1 + \delta_2$ holds. Therefore there is a δ, where $\varepsilon < \delta < \delta_1 + \delta_2$ and $\delta \geq \delta_1$. Since $1 - \delta_2 < 1 - (\delta - \delta_1)$ and αk is dense, there exists a s, such that $\alpha s \pmod 1 \in (1 - \delta_2, 1 - (\delta - \delta_1))$.

Now it will be shown, that $b_{s+1} \cdots b_{s+n} = b_{s+n+1} \cdots b_{s+2n}$ holds: First, for $1 \leq i \leq n$,

$$\alpha(s+n+i) \pmod 1 = (\alpha(s+i)+\varepsilon) \pmod 1$$

holds. Since $(\alpha(s+i)) \pmod 1 \notin [\frac{1}{2}-\varepsilon, \frac{1}{2}]$ it follows

$$(\alpha(s+i)+\tfrac{1}{2}) \pmod 1 \notin [1-\varepsilon, 1].$$

Let i be chosen arbitrarily ($1 \leq i \leq n$). Let be

$$n_1 = [\alpha(s+i-1)+\tfrac{1}{2}] \quad \text{and} \quad n_2 = [\alpha(s+n+i-1)+\tfrac{1}{2}].$$

If $b_{s+i} = 1$, then

$$\alpha(s+i)+\tfrac{1}{2} \in (n_1+1, n_1+2) \quad \text{and} \quad \alpha(s+n+i)+\tfrac{1}{2} \in (n_2+1, n_2+2).$$

Therefore $b_{s+n+i} = 1$ holds. If $b_{s+i} = 0$, then $\alpha(s+i)+\frac{1}{2} \in (n_1, n_1+1-\varepsilon)$ and therefore $\alpha(s+n+i)+\frac{1}{2} \in (n_2+\varepsilon, n_2+1)$; from this $b_{s+n+i} = 0$.

Remark. In a very similar way one can show the following: Let γ be an arbitrary real number, then the sequence $b'_1 b'_2 b'_3 \cdots$ where $b'_n = [\alpha n+\gamma]-[\alpha(n-1)+\gamma]$, is a sequence of unbounded repetition.

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